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INTEGRAL TRANSFORMS IN MATHEMATICAL PHYSICS

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by

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INTRODUCTION

THE classical methods of solution of the boundary-value problems of mathematical physics may be said to be derived from Fourier's pioneer work. Another technique, that of integral transforms, which had its origin in Heaviside's work, has been developed during the last few years and has certain advantages over the classical method. It is the purpose of this monograph to give an outline of the use of integral transforms in obtaining solutions to problems governed by *partial* differential equations with assigned boundary and initial conditions.

Heaviside (about 1890) was originally interested in the solution of the *ordinary* differential equations with constant coefficients occurring in the theory of electric circuits. Later he extended his method to the *partial* differential equations occurring in problems of electromagnetism and heat conduction. The power of his method was such that he solved many hitherto intractable problems and obtained solutions to problems already solved in a form better adapted for numerical computation. Later investigations by Bromwich, Carson and van der Pol placed the Heaviside calculus on a sound foundation.

The theory developed by Heaviside, Bromwich and Carson has been unified in recent work by Doetsch (and others) on the Laplace transformation. The solution found from Heaviside's calculus is obtained from Laplace's integral equation and the contour integral appearing in Bromwich's work is the integral in the inversion theorem for the Laplace transform.

The use of an integral transform will often reduce a partial differential equation in n independent variables to one in $(n-1)$ variables, thus reducing the difficulty of the problem under discussion. Successive operations of this type can sometimes ultimately reduce the problem to the solution of an ordinary differential equation, the theory of which has been extensively developed. Successive operations could, in fact, reduce the problem to the solution of an ordinary algebraic equation, but this is only sometimes worth while.

Although the Laplace transform has been more widely used, and is particularly suitable for problems governed by ordinary differential equations and for problems in heat conduction, other integral transforms can be very useful in the solution of the boundary-value problems of mathematical physics. Several different integral transforms so far have been successfully used in the solution of this type of problem, and there is no reason why the method should not be extended by the use of other kernels.

As already stated, the aim of the present monograph is to outline the procedure to be followed in using an integral transform in the solution of boundary-value problems. It is proposed also to show that a similar technique can be employed whatever the kernel or range of integration of the transform. Once the technique has been mastered, this method of solution is really more direct and straightforward than the classical method, which often demands great ingenuity in assuming at the outset the correct form for the solution. The technique can be reduced almost to a " drill ".

The point of view adopted is one which may perhaps be regarded as adequate for most investigations in mathematical physics. In applying an integral transform in this type of work, assumptions as to the commutability of certain limiting operations have to be made and often the derivation of the solution is not rigorous. Strictly speaking, it should be verified that the solution obtained by a purely formal procedure does in fact satisfy the differential equation and its boundary conditions. It is usually possible to do this, although in some cases the verification process is somewhat laborious.

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CHAPTER I

INTEGRAL TRANSFORMS AND THEIR INVERSION FORMULAE

1.1. We define the integral transform $\bar{f}(p)$ of a function $f(x)$ by the integral equation

$$\bar{f}(p)=\int_a^b f(x)K(p, x)\, dx, \quad . \quad . \quad . \quad (1.1)$$

$K(p, x)$ being a known function of p and x, called the *kernel* of the transform. When the limits a, b are both finite we shall speak of $\bar{f}(p)$ as the *finite* transform of $f(x)$. Such transforms and their applications are discussed in Chapter VI. At present we shall consider only transforms in which a is zero and b infinite except for the case of the transform given in (1.4) below where both limits are infinite.

In the application of integral transforms to the solution of boundary-value problems, use has so far been made of five different kernels. The general method could doubtless be extended by the use of other kernels. The five transforms considered here are :—

Laplace transform.

$$\bar{f}(p)=\int_0^\infty f(x)e^{-px}\, dx. \quad . \quad . \quad . \quad (1.2)$$

Fourier sine and cosine transforms.

$$\bar{f}(p)=\int_0^\infty f(x) \begin{matrix}\sin\\ \cos\end{matrix}\, px\, dx. \quad . \quad . \quad (1.3)$$

Complex Fourier transform.

$$\bar{f}(p)=\int_{-\infty}^\infty f(x)e^{ipx}\, dx. \quad . \quad . \quad . \quad (1.4)$$

Hankel transform.

$$\bar{f}(p)=\int_0^\infty f(x)xJ_n(px)\,dx, \quad . \quad . \quad . \quad (1.5)$$

where $J_n(px)$ is the Bessel function of the first kind of order n.

Mellin transform.

$$\bar{f}(p)=\int_0^\infty f(x)x^{p-1}\,dx. \quad . \quad . \quad . \quad . \quad (1.6)$$

As will be seen later, the effect of applying an integral transform to a partial differential equation is to exclude temporarily a chosen independent variable and to leave for solution a partial differential equation in one less variable. The solution of this equation will be a function of p and the remaining variables. When this solution has been obtained, it has to be "inverted" to recover the "lost" variable: thus if x is the variable eliminated and $\bar{f}(p)$ is one of the transforms given above, we first obtain auxiliary equations giving $\bar{f}$ in terms of p and the remaining independent variables, solve for $\bar{f}$ and then invert to obtain $f(x)$.

The inversion process means, in effect, the solution of one of the integral equations (1.2), . . . , (1.6), $\bar{f}(p)$ being supposed known and $f(x)$ to be found. Such solutions are known and can be obtained formally from Fourier's integral theorem which is given in the next paragraph.

1.2. FOURIER'S INTEGRAL FORMULA

As a preliminary to obtaining the inversion formulae, we shall require what is often known as Fourier's integral formula. A formal derivation is given below.

Suppose a function $f(x)$, of period $2\pi\lambda$, is given by the Fourier series

$$f(x)=\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}[a_n\cos(nx/\lambda)+b_n\sin(nx/\lambda)].$$

The coefficients a_0, a_n, b_n are obtained by multiplying successively by unity, $\cos(nx/\lambda)$, $\sin(nx/\lambda)$ and integrating with respect to x between $-\pi\lambda$, $\pi\lambda$. Since the circular functions are orthogonal, that is,

$$\int_{-\pi\lambda}^{\pi\lambda} \begin{matrix}\sin\\ \cos\end{matrix} \frac{nx}{\lambda}\, dx = 0,$$

$$\int_{-\pi\lambda}^{\pi\lambda} \begin{matrix}\sin\\ \cos\end{matrix} \frac{nx}{\lambda} \begin{matrix}\cos\\ \sin\end{matrix} \frac{mx}{\lambda}\, dx = 0,$$

$$\int_{-\pi\lambda}^{\pi\lambda} \begin{matrix}\sin\\ \cos\end{matrix} \frac{nx}{\lambda} \begin{matrix}\sin\\ \cos\end{matrix} \frac{mx}{\lambda}\, dx \left.\begin{matrix}=0, & m \neq n,\\ =\pi\lambda, & m=n,\end{matrix}\right\}$$

this process yields

$$\pi\lambda a_0 = \int_{-\pi\lambda}^{\pi\lambda} f(x')\, dx',$$

$$\pi\lambda a_n = \int_{-\pi\lambda}^{\pi\lambda} f(x') \cos(nx'/\lambda)\, dx',$$

$$\pi\lambda b_n = \int_{-\pi\lambda}^{\pi\lambda} f(x') \sin(nx'/\lambda)\, dx'.$$

Hence,

$$f(x) = \frac{1}{2\pi\lambda}\int_{-\pi\lambda}^{\pi\lambda} f(x')\, dx' + \frac{1}{\pi\lambda}\sum_{n=1}^{\infty}\int_{-\pi\lambda}^{\pi\lambda} f(x') \cos\frac{n(x-x')}{\lambda}\, dx'.$$

Putting $n/\lambda = \alpha$, $1/\lambda = \delta\alpha$ and making λ tend to infinity, the sum passes formally into an integral and we have

$$\pi f(x) = \int_0^\infty d\alpha \int_{-\infty}^{\infty} f(x') \cos\alpha(x-x')\, dx', \qquad (1.7)$$

which is Fourier's integral formula.

It is emphasised that the analysis given above is purely formal. For a rigorous discussion and a precise statement of the conditions under which (1.7) holds, reference should be made to a standard text.* It is sufficient that

* See, for example, E. C. Titchmarsh, *Theory of Fourier Integrals*, Oxford, (1937), § 1.9.

$f(x)$ should satisfy Dirichlet's conditions* in any finite interval and that $\int_{-\infty}^{\infty} f(x)\,dx$ should be absolutely convergent. Common types of function which satisfy Dirichlet's conditions are functions possessing only a finite number of maxima, minima and ordinary discontinuities or functions of bounded variation in any finite interval.

1.3. INVERSION FORMULAE

Inversion formulae (solutions of the integral equations (1.2), . . . , (1.6)) are now formally derived from Fourier's integral formula. That conditions must be imposed on $f(x)$ and on the paths of integration of the contour integrals appearing in some of the formulae will be clear from the examples given. Any attempt to establish these formulae by rigorous analysis would be somewhat out of place here—in the type of work in which integral transforms are here to be applied, the necessary conditions are almost always fulfilled in the physical problem under discussion.

(i). *Inversion formula for the Laplace transform.*

The repeated integral in Fourier's integral formula (1.7) can be written

$$\tfrac{1}{2}\int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} f(x') \cos \alpha(x-x')\,dx',$$

and it is clear that

$$\int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} f(x') \sin \alpha(x-x')\,dx' = 0.$$

Thus formula (1.7) can be written in the form

$$2\pi f(x) = \int_{-\infty}^{\infty} e^{i\alpha x}\,d\alpha \int_{-\infty}^{\infty} f(x')e^{-i\alpha x'}\,dx'. \quad . \quad (1.8)$$

* See, for example, H. S. Carslaw, *Fourier Series and Integrals*, Macmillan, 2nd Ed., (1921), § 93.

Using (1.2), the definition of the Laplace transform, we have

$$\int_{\gamma-i\omega}^{\gamma+i\omega} e^{xp}\bar{f}(p)\,dp=\int_{\gamma-i\omega}^{\gamma+i\omega} e^{xp}\,dp\int_0^\infty f(x')e^{-px'}\,dx'$$

$$=ie^{\gamma x}\int_{-\omega}^{\omega} e^{ixy}\,dy\int_0^\infty e^{-iyx'}[e^{-\gamma x'}f(x')]\,dx', \quad (1.9)$$

on writing $p=\gamma+iy$. But the limit as ω tends to infinity of the double integral on the right-hand side of equation (1.9) is, by (1.8), equal to $2\pi e^{-\gamma x}f(x)$ for $x>0$ and to zero for $x<0$. Hence (1.9) gives

$$2\pi i f(x)=\int_{\gamma-i\infty}^{\gamma+i\infty} e^{xp}\bar{f}(p)\,dp, \quad (1.10)$$

for $x>0$ and zero for $x<0$.

Equation (1.10) is the required inversion formula for the Laplace transform. Clearly, conditions must be imposed on $f(x)$ for the integral in (1.2), defining the Laplace transform, to exist. It is also necessary, as we shall see by the example given below, for the line of integration of the integral in (1.10) to have limitations as to its position in the plane of the complex variable p. It is, in fact, necessary that γ shall be greater than the real parts of all the singularities of $\bar{f}(p)$.

Example.

If $f(x)=e^{-x}$, its Laplace transform is given by

$$\bar{f}(p)=\int_0^\infty e^{-x}e^{-px}\,dx$$

$$=\int_0^\infty e^{-(p+1)x}\,dx=(p+1)^{-1}.$$

In this example it is necessary to take the real part of p to be greater than -1 for $\bar{f}(p)$ to exist.

Suppose now we wish to carry out the inverse process; that is, we wish to find the function $f(x)$ whose Laplace transform is $\bar{f}(p)=(p+1)^{-1}$, the real part of p being greater than -1. The inversion formula (1.10) gives

$$2\pi i f(x)=\int_{\gamma-i\infty}^{\gamma+i\infty} e^{xp}(p+1)^{-1}\,dp.$$

The line of integration is parallel to the imaginary axis in the p-plane and must lie on the right of the singularities of the integrand, in this case, a simple pole at $p=-1$. The integral is easily evaluated by completing the contour by the arc of a circle, centre the origin, and lying to the left of the line of integration. On this circular arc $p=Re^{i\theta}$, $\pi/2<\theta<3\pi/2$, and it is easy to show that the integral along the arc tends to zero as R tends to infinity. Cauchy's theorem now shows that the line integral is equal to $2\pi i$ times the residue at the pole $p=-1$, that is,

$$f(x)=e^{-x},$$

as we should expect.

(ii). *Inversion formulae for Fourier sine and cosine transforms.*

Fourier's integral formula (1.7) can be written (with $\alpha=p$) in the form

$$\pi f(x)=\int_0^\infty\left\{\int_{-\infty}^\infty f(x')\cos px'\,dx'\right\}\cos xp\,dp$$
$$+\int_0^\infty\left\{\int_{-\infty}^\infty f(x')\sin px'\,dx'\right\}\sin xp\,dp. \quad (1.11)$$

If $f(x)$ is an odd function of x,

$$\int_{-\infty}^\infty f(x')\cos px'\,dx'=0$$

and

$$\int_{-\infty}^\infty f(x')\sin px'\,dx'=2\int_0^\infty f(x')\sin px'\,dx'$$
$$=2\bar{f}(p),$$

where $\bar{f}(p)$ is the Fourier sine transform of $f(x)$ defined in equation (1.3). Equation (1.11) gives in this case

$$f(x)=(2/\pi)\int_0^\infty \bar{f}(p)\sin xp\,dp, \quad . \quad . \quad (1.12)$$

which is an inversion formula for the sine transform.

Similarly if $f(x)$ is an even function of x,

$$\int_{-\infty}^{\infty} f(x')\sin px'\,dx'=0$$

and

$$\int_{-\infty}^{\infty} f(x')\cos px'\,dx'=2\int_0^\infty f(x')\cos px'\,dx'$$
$$=2\bar{f}(p),$$

where $\bar{f}(p)$ is now the Fourier cosine transform, also defined in equation (1.3). Substitution in (1.11) now gives, as the inversion formula for the cosine transform,

$$f(x)=(2/\pi)\int_0^\infty \bar{f}(p)\cos xp\,dp. \quad . \quad . \quad (1.13)$$

Examples.

If $f(x)=e^{-x}$, the sine transform is

$$\int_0^\infty e^{-x}\sin px\,dx=p/(1+p^2),$$

while the cosine transform is

$$\int_0^\infty e^{-x}\cos px\,dx=1/(1+p^2).$$

If we are given

$$\bar{f}(p)=p/(1+p^2)$$

as the sine transform of $f(x)$, the inversion formula (1.12) gives

$$f(x)=\frac{2}{\pi}\int_0^\infty \frac{p}{1+p^2}\sin xp\,dp=e^{-x};$$

while if

$$\bar{f}(p)=1/(1+p^2)$$

is given as the cosine transform of $f(x)$, formula (1.13) gives

$$f(x)=\frac{2}{\pi}\int_0^\infty \frac{\cos xp}{1+p^2}\,dp=e^{-x},$$

so that, in each case, we recover the original function e^{-x}, as we should.

(iii). *Inversion formulae for the complex Fourier transform.*

We have already seen, equation (1.8), that Fourier's integral formula can be written in the form

$$2\pi f(x)=\int_{-\infty}^{\infty} e^{i\alpha x}\,d\alpha\int_{-\infty}^{\infty} f(x')e^{-i\alpha x'}\,dx'.$$

Writing $\alpha=-p$ and remembering that the complex Fourier transform is given by (1.4), viz.,

$$\bar{f}(p)=\int_{-\infty}^{\infty} f(x)e^{ipx}\,dx, \qquad . \quad . \quad . \quad (1.14)$$

we have

$$2\pi f(x)=\int_{-\infty}^{\infty} \bar{f}(p)e^{-ixp}\,dp, \qquad . \quad . \quad (1.15)$$

which can be regarded as the required inversion formula.

Example.

If $f(x)=e^{-|x|}$, the complex Fourier transform is

$$\bar{f}(p)=\int_0^\infty e^{-(1-ip)x}\,dx+\int_{-\infty}^0 e^{(1+ip)x}\,dx$$
$$=(1-ip)^{-1}+(1+ip)^{-1}=2/(1+p^2).$$

If $\bar{f}(p)=2/(1+p^2)$, the inversion formula (1.15) gives

$$\pi f(x)=\int_{-\infty}^{\infty} \frac{e^{-ixp}}{1+p^2}\,dp.$$

To evaluate this integral we note that, for $x\geqslant 0$, the integral taken round a semi-circle, centre the origin and lying below the real axis in the p-plane, tends to zero as the radius of the circle tends to infinity. The integral can therefore be replaced by $-2\pi i$ times the residue at the only pole (at $p=-i$) within this contour, the negative sign being affixed since the direction round the contour is clockwise. Thus

$$f(x)=-2i\,e^{-ix(-i)}/[(d/dp)(1+p^2)]_{p=-i}$$
$$=e^{-x}.$$

Similarly, for $x\leqslant 0$, we close the contour by a semi-circle lying above the real axis and obtain, from the residue at the pole at $p=i$,

$$f(x)=e^{x}.$$

Both results are included in $f(x)=e^{-|x|}$.

The existence of $\bar{f}(p)$ as given by (1.14) imposes severe restrictions upon the form of $f(x)$. For instance, if $f(x)=e^{-x}$, the integral in (1.14) does not exist.

Even if $\bar{f}(p)$ does not exist, the functions given by

$$\left.\begin{aligned}\bar{f}_+(p)&=\int_0^\infty f(x)e^{ipx}\,dx,\\ \bar{f}_-(p)&=\int_{-\infty}^0 f(x)e^{ipx}\,dx,\end{aligned}\right\} \quad . \quad . \quad . \quad (1.16)$$

where $p=u+iv$, may exist; thus $\bar{f}_+(p)$ may exist for a sufficiently large positive value of v (say a) while $\bar{f}_-(p)$ may exist for a sufficiently large negative value of v (say b). If this is the case

$$\bar{f}_+(p)=\int_0^\infty f(x)e^{-ax}e^{iux}\,dx, \quad . \quad . \quad (1.17)$$

so that $\bar{f}_+(p)$ is the transform of a function equal to

$$f(x)e^{-ax} \text{ for } x>0$$

and to zero for $x<0$.

Hence, from (1.15),

$$\int_{-\infty}^\infty \bar{f}_+(p)e^{-ixu}\,du=2\pi f(x)e^{-ax},\ x>0,$$
$$=0,\ x<0,$$

giving

$$\int_{-\infty}^{\infty} \bar{f}_+(p) e^{-ix(u+ia)}\, du = 2\pi f(x),\ x>0,$$

$$=0,\ x<0.$$

Similarly

$$\int_{-\infty}^{\infty} \bar{f}_-(p) e^{-ix(u+ib)}\, du = 0,\ x>0,$$

$$=2\pi f(x),\ x<0.$$

Writing $u=p-ia$ in the integral containing $\bar{f}_+(p)$ and $u=p-ib$ in the integral containing $\bar{f}_-(p)$ and adding, we have

$$2\pi f(x) = \int_{ia-\infty}^{ia+\infty} \bar{f}_+(p) e^{-ixp}\, dp + \int_{ib-\infty}^{ib+\infty} \bar{f}_-(p) e^{-ixp}\, dp, \quad (1.18)$$

as the appropriate inversion formula.

As in the case of the path of integration for the contour integral in the inversion formula for the Laplace transform, a and b are related to the singularities of $\bar{f}_+(p)$ and $\bar{f}_-(p)$. It is necessary that a shall be greater than the imaginary parts of all singularities of $\bar{f}_+(p)$ and that b shall be less than the imaginary parts of all singularities of $\bar{f}_-(p)$.

Example.

If $f(x)=e^{-x}$,

$$\bar{f}_+(p) = \int_0^{\infty} e^{-x} e^{ipx}\, dx = (1-ip)^{-1},$$

provided a (the imaginary part of p) is greater than -1, and

$$\bar{f}_-(p) = \int_{-\infty}^{0} e^{-x} e^{ipx}\, dx = -(1-ip)^{-1},$$

provided b is less than -1.

Taking these values of $\bar{f}_+(p)$ and $\bar{f}_-(p)$, the inversion formula (1.18) gives

$$2\pi f(x)=\int_{ia-\infty}^{ia+\infty}\frac{e^{-ixp}}{1-ip}\,dp-\int_{ib-\infty}^{ib+\infty}\frac{e^{-ixp}}{1-ip}\,dp.$$

Take the case $x>0$ and consider the first line integral. Here $a>-1$, and the integral round the arc of a circle, centre the origin and lying below the real axis, can be shown to tend to zero as the radius of the arc tends to infinity. The closed contour contains a pole at $p=-i$, and the first integral is equal to $2\pi e^{-x}$. The second integral is zero because $b<-1$ and the contour contains no pole. A similar discussion shows that $f(x)=e^{-x}$ also for the case $x<0$.

(iv). *Inversion formula for the Hankel transform.*

The results (1.14), (1.15) for the complex Fourier transform can be extended to cover functions of two variables. Thus if

$$\left.\begin{aligned}\bar{f}(s,t)&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)\exp[i(sx+ty)]\,dx\,dy,\\ &\text{then}\\ 4\pi^2 f(x,y)&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\bar{f}(s,t)\exp[-i(xs+yt)]\,ds\,dt.\end{aligned}\right\}\quad(1.19)$$

Writing

$$x=r\cos\theta,\quad y=r\sin\theta,\quad s=p\cos\alpha,\quad t=p\sin\alpha,$$

the theorem given by equations (1.19) becomes :—

if

$$\left.\begin{aligned}\bar{f}(p,\alpha)&=\int_0^{\infty}r\,dr\int_0^{2\pi}f(r,\theta)\exp[ipr\cos(\theta-\alpha)]\,d\theta,\\ &\text{then}\\ 4\pi^2 f(r,\theta)&=\int_0^{\infty}p\,dp\int_0^{2\pi}\bar{f}(p,\alpha)\exp[-irp\cos(\theta-\alpha)]\,d\alpha.\end{aligned}\right\}(1.20)$$

Take for $f(r, \theta)$ the special function $f(r)e^{-in\theta}$, then the first of equations (1.20) gives

$$\bar{f}(p, \alpha)=\int_0^\infty f(r)r\,dr\int_0^{2\pi} \exp\,[i\{-n\theta+pr\cos(\theta-\alpha)\}]d\theta. \quad (1.21)$$

By writing $\phi=\alpha-\theta-\pi/2$, the integral in θ can be written

$$\exp\,[in(\pi/2-\alpha)]\int_0^{2\pi} \exp\,[i\{n\phi-pr\sin\phi\}]\,d\phi$$

$$=2\pi\exp\,[in(\pi/2-\alpha)]J_n(pr),$$

for the integral in ϕ is Bessel's integral.*

If we denote the Hankel transform of $f(r)$ by $\bar{f}(p)$ so that, by equation (1.5),

$$\bar{f}(p)=\int_0^\infty f(r)rJ_n(pr)\,dr, \quad . \quad . \quad . \quad (1.22)$$

equation (1.21) can be written

$$\bar{f}(p, \alpha)=2\pi\exp\,[in(\pi/2-\alpha)]\bar{f}(p). \quad . \quad . \quad (1.23)$$

Substituting $f(r)e^{-in\theta}$ for $f(r, \theta)$ and the expression given in equation (1.23) for $\bar{f}(p, \alpha)$ in the second of equations (1.20), we have

$$2\pi f(r)e^{-in\theta}$$

$$=\int_0^\infty p\bar{f}(p)\,dp\int_0^{2\pi}\exp\,[i\{n(\pi/2-\alpha)-pr\cos(\theta-\alpha)\}]\,d\alpha.$$

By writing $\phi=\theta-\alpha+\pi/2$, the integral in α can be written

$$e^{-in\theta}\int_0^{2\pi}\exp\,[i(n\phi-pr\sin\phi)]\,d\phi.$$

Again using Bessel's integral, this can be expressed as

$$2\pi e^{-in\theta}J_n(rp).$$

Hence

$$f(r)=\int_0^\infty \bar{f}(p)pJ_n(rp)\,dp, \quad . \quad . \quad . \quad (1.24)$$

which is the required inversion formula for the Hankel transform defined in (1.22).

* G. N. Watson, *Bessel Functions*, Cambridge, (1944), § 2.2.

Example.

As a simple example, take $f(r)=1/r$.

Then the Hankel transform is given by

$$\bar{f}(p)=\int_0^\infty J_n(pr)\,dr=1/p.$$

If we are given $\bar{f}(p)=1/p$, the inversion formula (1.24) gives

$$f(r)=\int_0^\infty J_n(rp)\,dp=1/r,$$

as, of course, it should.

(v). *Inversion formula for the Mellin transform.*

The Mellin transform, defined in (1.6), is

$$\bar{f}(p)=\int_0^\infty f(x)x^{p-1}\,dx.$$

Writing $x=e^\xi$, this becomes

$$\bar{f}(p)=\int_{-\infty}^\infty f(e^\xi)e^{p\xi}\,d\xi$$

$$=\int_{-\infty}^\infty f(e^\xi)e^{\gamma\xi}e^{i\omega\xi}\,d\xi, \quad . \quad . \quad . \quad (1.25)$$

where $p=\gamma+i\omega$.

From equation (1.14), $\bar{f}(p)$ is the Fourier transform of $f(e^\xi)e^{\gamma\xi}$ and the inversion formula (1.15) gives

$$2\pi f(e^\xi)e^{\gamma\xi}=\int_{-\infty}^\infty \bar{f}(p)e^{-i\xi\omega}\,d\omega$$

$$=-i\int_{\gamma-i\infty}^{\gamma+i\infty}\bar{f}(p)e^{-\xi(p-\gamma)}\,dp,$$

since $i\omega=p-\gamma$.

In terms of x, the required inversion formula is

$$2\pi i f(x)=\int_{\gamma-i\infty}^{\gamma+i\infty}\bar{f}(p)x^{-p}\,dp, \quad x>0, \quad . \quad (1.26)$$

and γ has to be chosen so that the integral in (1.25) exists.

Once again, the treatment given here has been purely formal and no attempt has been made to establish rigorously conditions for validity.

Example.

Take $f(x)=e^{-x}$, then the Mellin transform is given by

$$\bar{f}(p)=\int_0^\infty e^{-x}x^{p-1}\,dx=\Gamma(p),$$

where $\Gamma(p)$ is the gamma-function of argument p.

If we are given $\bar{f}(p)=\Gamma(p)$, the inversion formula (1.26) gives

$$2\pi i f(x)=\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma(p)x^{-p}\,dp$$

$$=\pi\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{x^{-p}\,dp}{\Gamma(1-p)\sin\pi p},$$

for

$$\Gamma(p)\Gamma(1-p)=\pi/\sin\pi p.$$

If the contour is closed by the arc of a circle, centre the origin and lying to the left of the imaginary axis, the integral round this arc can be shown to tend to zero as the radius of the arc tends to infinity. The line integral can therefore be replaced by $2\pi i$ times the sum of the residues at the poles at $p=0, -1, -2, \ldots$

Hence

$$f(x)=\pi\sum_{n=0}^{\infty}x^n \Big/ \left[\Gamma(1+n)\frac{d}{dp}(\sin\pi p)\right]_{p=-n}$$

$$=\sum_{n=0}^{\infty}(-1)^n x^n/n!=e^{-x}.$$

1.4. SUMMARY

The five integral transforms and their inversion formulae are collected on page 15 for easy reference.

(i). *Laplace transform.*

$$\bar{f}(p)=\int_0^\infty f(x)e^{-px}\,dx, \quad . \quad . \quad . \quad (1.2)$$

$$2\pi i f(x)=\int_{\gamma-i\infty}^{\gamma+i\infty}\bar{f}(p)e^{xp}\,dp, \quad . \quad . \quad (1.10)$$

where γ is greater than the real parts of all singularities of $\bar{f}(p)$.

(ii). *Fourier sine and cosine transforms.*

$$\bar{f}(p)=\int_0^\infty f(x)\,\begin{matrix}\sin\\ \cos\end{matrix}\,px\,dx, \quad . \quad . \quad . \quad (1.3)$$

$$f(x)=\frac{2}{\pi}\int_0^\infty \bar{f}(p)\,\begin{matrix}\sin\\ \cos\end{matrix}\,xp\,dp. \quad (1.12), (1.13)$$

(iii). *Complex Fourier transform.*

$$\bar{f}(p)=\int_{-\infty}^\infty f(x)e^{ipx}\,dx, \quad . \quad . \quad . \quad (1.4)$$

$$2\pi f(x)=\int_{-\infty}^\infty \bar{f}(p)e^{-ixp}\,dp. \quad . \quad . \quad (1.15)$$

A more general form is

$$\left.\begin{aligned}\bar{f}_+(p)&=\int_0^\infty f(x)e^{ipx}\,dx,\\ \bar{f}_-(p)&=\int_{-\infty}^0 f(x)e^{ipx}\,dx,\end{aligned}\right\} \quad . \quad . \quad . \quad (1.16)$$

in which case

$$2\pi f(x)=\int_{ia-\infty}^{ia+\infty}\bar{f}_+(p)e^{-ixp}\,dp+\int_{ib-\infty}^{ib+\infty}\bar{f}_-(p)e^{-ixp}\,dp, \quad (1.18)$$

where a is greater than the imaginary parts of all singularities of $\bar{f}_+(p)$ and b is less than the imaginary parts of all singularities of $\bar{f}_-(p)$.

(iv). *Hankel transform.*

$$\bar{f}(p)=\int_0^\infty f(r)rJ_n(pr)\,dr, \quad . \quad . \quad . \quad (1.5)$$

$$f(r)=\int_0^\infty \bar{f}(p)pJ_n(rp)\,dp. \quad . \quad . \quad (1.24)$$

(v). *Mellin transform.*

$$\bar{f}(p)=\int_0^\infty f(x)x^{p-1}\,dx, \quad . \quad . \quad . \quad (1.6)$$

$$2\pi i f(x)=\int_{\gamma-i\infty}^{\gamma+i\infty} \bar{f}(p)x^{-p}\,dp. \quad . \quad . \quad (1.26)$$

EXAMPLES ON CHAPTER I

1. (*a*) Show that the Laplace transform of $\sin \omega t$ is $\omega/(p^2+\omega^2)$.

(*b*) Use the inversion formula to show that $p/(p^2+\omega^2)$ is the Laplace transform of $\cos \omega t$.

2. From the integral

$$\int_0^\infty \exp\left[-\alpha^2x^2-\beta^2x^{-2}\right]dx=(\sqrt{\pi}/2\alpha)e^{-2\alpha\beta},$$

α, β positive, show that $e^{-a\sqrt{p}}/\sqrt{p}$ is the Laplace transform of $(1/\pi^{\frac{1}{2}}t^{\frac{1}{2}})\exp(-a^2/4t)$, $a>0$.

3. If $\bar{f}(p)$ is the Laplace transform of $f(x)$ and $a>0$, show that $e^{-ap}\bar{f}(p)$ is the transform of $f(x-a)H(x-a)$ where $H(x)=0$, $x<0$, $H(x)=1$, $x>0$.

4. Find the cosine transform of a function of x which is unity for $0<x<a$ and zero for $x>a$.

What is the function whose cosine transform is $(\sin ap)/p$?

5. Show that the cosine transform of a function of x which is equal to $\cos x$ for $0<x<a$ and zero for $x>a$ is

$$\tfrac{1}{2}\left[\frac{\sin(1+p)a}{1+p}+\frac{\sin(1-p)a}{1-p}\right].$$

6. Use the result $J_{\frac{1}{2}}(z)=(2/\pi z)^{\frac{1}{2}} \sin z$ to deduce the Fourier sine transform and its inversion formula from the corresponding formulae for the Hankel transform.

7. Show that the Mellin transform of a function of x which is unity for $x<a$ and zero for $x \geqslant a$ is a^p/p, and that of a function which is equal to $\log (a/x)$ for $x<a$ and zero for $x \geqslant a$ is a^p/p^2.

CHAPTER II

THE LAPLACE TRANSFORM

2.1. The procedure to be followed in solving a differential equation with assigned boundary and initial conditions by the use of any integral transform is briefly as follows:

(i) Select the appropriate transform; guidance on this is given in the examples appearing in this and subsequent chapters.

(ii) Multiply the differential equation and boundary conditions by the selected kernel and integrate between appropriate limits with respect to the variable selected for exclusion.

(iii) In performing the integration in (ii) make use of the appropriate boundary (or initial) conditions in evaluating terms at the limits of integration.

(iv) Solve the resulting "auxiliary" equations, so obtaining the transform of the wanted function.

(v) Invert to obtain the wanted function itself.

The above procedure is illustrated in detail in subsequent paragraphs for the Laplace transform and in subsequent chapters for the other transforms defined in Chapter I.

2.2. THE APPLICATION OF THE LAPLACE TRANSFORM TO ORDINARY DIFFERENTIAL EQUATIONS

The principal aim of the present book is to discuss the solution of partial differential equations by the use of integral transforms. A detailed discussion of their use in the solution of ordinary differential equations is therefore rather out of place. Very adequate accounts have been given by Carslaw, Jaeger and others.*

* See, for example, H. S. Carslaw and J. C. Jaeger, *Operational Methods in Applied Mathematics*, Oxford, (1941), or J. C. Jaeger, *Introduction to the Laplace Transformation with Engineering Applications*, Methuen, (1948).

It is, however, worth while to illustrate the technique by a single simple example in ordinary differential equations. Suppose it is required to find the electric current I at time t in a circuit consisting of inductance L and resistance R when a constant electromotive force E is applied at time $t=0$. The current is given by the ordinary first order differential equation

$$L(dI/dt)+RI=E, \quad . \quad . \quad . \quad (2.1)$$

with the initial condition

$$I=0, \text{ when } t=0. \quad . \quad . \quad . \quad . \quad (2.2)$$

For an equation of the first order with the range $(0, \infty)$ for the independent variable, the Laplace transform is appropriate. We therefore multiply equation (2.1) by e^{-pt} and integrate with respect to t from 0 to ∞. Integration by parts gives

$$\int_0^\infty e^{-pt}(dI/dt)\,dt=\Big[e^{-pt}I\Big]_{t=0}^{\infty}+p\int_0^\infty e^{-pt}I\,dt. \quad (2.3)$$

The first term on the right-hand side vanishes at the upper limit since the exponential term vanishes there. It vanishes also at the lower limit through the term I, which, by the initial condition (2.2), is zero at $t=0$. If we denote by $\bar{I}$ the Laplace transform of I, so that

$$\bar{I}=\int_0^\infty Ie^{-pt}\,dt, \quad . \quad . \quad . \quad . \quad (2.4)$$

equation (2.3) gives

$$\int_0^\infty e^{-pt}(dI/dt)\,dt=p\bar{I}. \quad . \quad . \quad . \quad (2.5)$$

The operation of multiplying equation (2.1) by the kernel of the Laplace transform and integration with respect to t between 0 and ∞ leads therefore, when we make use of (2.4) and (2.5), to the algebraic equation

$$Lp\bar{I}+R\bar{I}=E/p, \quad . \quad . \quad . \quad . \quad (2.6)$$

for $E\int_0^\infty e^{-pt}dt=E/p$. This is the auxiliary equation for this simple problem, the independent variable t of the original equation having been excluded. Solving for $\bar{I}$ we have

$$\bar{I}=Ep^{-1}(Lp+R)^{-1}$$
$$=\frac{E}{R}\left[\frac{1}{p}-\frac{1}{p+R/L}\right]. \quad \quad (2.7)$$

We now have to invert to obtain I from $\bar{I}$. This can always be done from the inversion formula (1.10), but for this type of problem the different functions which arise in practice are few enough to list in a table of transforms.

Short Table of Laplace Transforms

$$\bar{f}(p)=\int_0^\infty e^{-pt}f(t)\,dt.$$

$\bar{f}(p)$	$f(t)$
p^{-n-1}	$t^n/\Gamma(n+1),\ n>-1.$
$(p+\alpha)^{-1}$	$e^{-\alpha t}.$
$\omega/(p^2+\omega^2)$	$\sin \omega t.$
$p/(p^2+\omega^2)$	$\cos \omega t.$
$(e^{-a\sqrt{p}})/p$	$\operatorname{erfc}(a/2\sqrt{t}) =(2/\sqrt{\pi})\int_{a/2\sqrt{t}}^\infty e^{-u^2}\,du,\ a>0.$
$(e^{-a\sqrt{p}})/p^{3/2}$	$2\sqrt{t}\operatorname{ierfc}(a/2\sqrt{t})$, where $\operatorname{ierfc} x=\int_x^\infty \operatorname{erfc} u\,du,\ a>0.$

Such tables are prepared from known definite integrals or by applying the inversion formula. A short table is given below. Very extensive tables (or dictionaries) are available, two of the largest being "*Tabellen zur Laplace-Transformation und Anleitung zum Gebrauch*", by G. Doetsch, Springer, Berlin, 1947, and "*Formulaire pour le Calcul symbolique*", by N. W. McLachlan and P. Humbert, Gauthier-Villars, Paris, 1941. It should be noted that in the French dictionary the fundamental transform is that used by Heaviside, viz.,

$$\bar{f}(p)=p\int_0^\infty e^{-px}f(x)\,dx,$$

and it differs from the Laplace transform by the factor p.

Using the table given below, we have

$$I=(E/R)[1-\exp(-Rt/L)] \quad . \quad . \quad (2.8)$$

as the solution to our present problem.

2.3. A SIMPLE PROBLEM IN HEAT CONDUCTION

As a first example of the application of the transform method to a problem governed by a partial differential equation, we consider the classical problem of heat flow in the semi-infinite solid $x>0$ when the boundary $x=0$ is kept at a constant temperature V_0, the initial temperature of the solid being zero.

If V is the temperature at time t and κ the diffusivity of the material, we have to find V from the partial differential equation

$$\frac{\partial V}{dt}=\kappa\frac{\partial^2 V}{\partial x^2}, \qquad x>0, \qquad t>0, \quad . \quad . \quad (2.9)$$

with the boundary condition

$$V=V_0, \text{ when } x=0, \quad t>0, \quad . \quad . \quad . \quad (2.10)$$

and the initial condition

$$V=0, \text{ when } t=0, \quad x>0. \quad . \quad . \quad . \quad (2.11)$$

Writing

$$\bar{V}=\int_0^\infty e^{-pt}V\,dt, \quad . \quad . \quad . \quad . \quad (2.12)$$

so that $\bar{V}$ is the Laplace transform of the temperature, we have by integration by parts

$$\int_0^\infty e^{-pt}(\partial V/\partial t)\,dt=\Big[e^{-pt}V\Big]_{t=0}^\infty+p\int_0^\infty e^{-pt}V\,dt=p\bar{V}, \quad (2.13)$$

using the initial condition (2.11) to evaluate the first term on the right-hand side at $t=0$. At the upper limit this term vanishes through the exponential factor.

Multiplying the differential equation (2.9) and the boundary condition (2.10) by the kernel e^{-pt} of the Laplace transform, integrating with respect to t between 0, ∞ and using (2.12), (2.13), we have the auxiliary equations

$$\kappa(d^2\bar{V}/dx^2)=p\bar{V}, \quad x>0, \quad . \quad . \quad (2.14)$$

with

$$\bar{V}=\int_0^\infty e^{-pt}V_0\,dt=V_0/p, \text{ when } x=0. \quad . \quad (2.15)$$

By this procedure we have reduced the problem to the solution of the *ordinary* differential equation (2.14). Since the temperature is finite as x tends to infinity, the solution of (2.14) satisfying the boundary condition (2.15) is

$$\bar{V}=(V_0/p)e^{-x\sqrt{(p/\kappa)}}. \quad . \quad . \quad . \quad (2.16)$$

Inversion to V is now accomplished by use of the inversion formula (1.10) or of a table of Laplace transforms. From the latter we have

$$V=V_0 \operatorname{erfc}\{x/2\sqrt{(\kappa t)}\}, \quad . \quad . \quad . \quad (2.17)$$

as the required solution, where erfc x is the complementary error function defined by

$$\operatorname{erfc} x=1-\operatorname{erf} x$$

$$=(2/\sqrt{\pi})\int_x^\infty e^{-u^2}\,du.$$

2.4. REMARKS ON THE EVALUATION OF THE CONTOUR INTEGRAL OF THE INVERSION FORMULA

In the solution of the boundary-value problems of mathematical physics, inversion by a table of transforms will generally only be possible if the table is a very extensive one. Even when a table of this sort is available there will be many occasions when $\bar{f}(p)$ will not appear in its contents. In such cases we shall be driven to the use of the inversion formula (1.10) and the following remarks may be useful for the evaluation of the contour integral.

The details of the calculation depend on the nature of the transform $\bar{f}(p)$ of the wanted function. $\bar{f}(p)$ is usually either:—

(i) A single-valued function of p with a finite or enumerably infinite number of poles in the complex plane, or

(ii) a function of p with a branch point at the origin and, possibly, a finite number of poles.

In the case (i) the contour is completed by an arc of a circle of radius R as shown in Fig. 1. We choose for R a sequence of values R_n so that the arc does not pass through a pole of $\bar{f}(p)$. Next we try to show that the integral taken round the arc tends to zero as R_n tends to infinity. In most problems of this type the integral does in fact vanish; a condition for this to be so has been given by Carslaw and Jaeger.* Briefly, the sufficient condition is that

$$|\bar{f}(p)| < CR^{-k}, \quad . \quad . \quad . \quad . \quad (2.18)$$

when

$$p = Re^{i\theta}, \quad -\pi \leqslant \theta \leqslant \pi, \quad R > R_0,$$

where R_0, C, k are constants and $k > 0$.

If this condition is fulfilled, the line integral in the inversion formula (1.10) can, by Cauchy's theorem, be replaced by the limit as n tends to infinity of $2\pi i$ times the

* H. S. Carslaw and J. C. Jaeger, *Operational Methods in Applied Mathematics*, Oxford, (1941), § 31.

sum of the residues of the integrand at its poles within the circle of radius R_n.

An example of this procedure has already been given on page 6 for the simple example $f(p)=1/(p+1)$, in which case there is a single pole at $p=-1$ and $f(x)$ turns out to be e^{-x}. Another example is given in § 2.5.

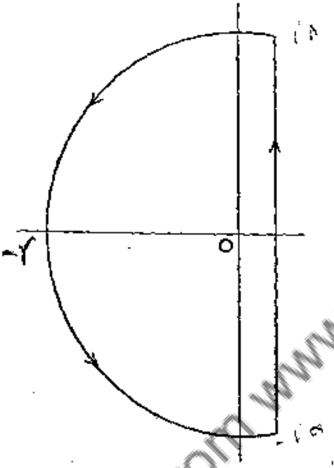

FIG. 1

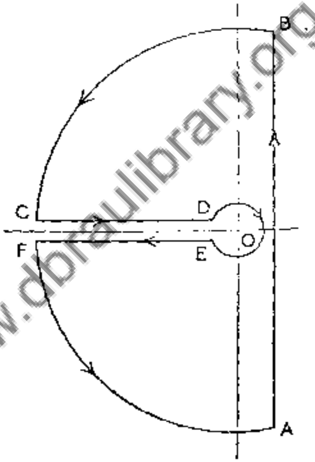

FIG. 2

When $\bar{f}(p)$ has a branch point at the origin, the contour is closed by the arcs BC, FA, a cut along the negative real axis CD, EF and a small circle surrounding the origin as shown in Fig. 2. If, as is usually the case, $\bar{f}(p)$ satisfies the condition (2.18), the integral round the circular arcs will tend to zero as the radii of the arcs tend to infinity. Since the integrand is single-valued inside and on the contour, the line integral can therefore be replaced by a real infinite integral obtained from the integrals along DC, FE, a term from the small circle surrounding the origin and $2\pi i$ times the residues at any poles inside the contour.

As an example of the use of the contour of Fig. 2 we will invert the function $(e^{-a\sqrt{p}})/p$, this being, apart from a constant, the transform of the temperature in the heat conduction problem of § 2.3, when we write $a=x/\sqrt{\kappa}$.

By the inversion formula (1.10) we require to evaluate

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \exp\,(tp-a\sqrt{p})\frac{dp}{p}. \quad . \quad . \quad (2.19)$$

The integrand has a branch point at the origin and is single-valued inside and on the contour of Fig. 2. Since

$$|(e^{-a\sqrt{p}})/p|<|p|^{-1},$$

the condition (2.18) is satisfied and the limits of the integrals round the arcs BC, FA tend to zero as the radii of the arcs tend to infinity. The line integral of (2.19) is therefore equal to the sum of the integrals along DC, FE and the limit of that over the small circle as its radius tends to zero, the integrand having no poles inside the contour.

On DC, we write $p=\alpha e^{i\pi}$ and obtain a contribution

$$\frac{1}{2\pi i}\int_0^\infty e^{-\alpha t}e^{-ia\sqrt{\alpha}}\,\frac{d\alpha}{\alpha}.$$

Similarly on FE, we put $p=\alpha e^{-i\pi}$ and get

$$-\frac{1}{2\pi i}\int_0^\infty e^{-\alpha t}e^{ia\sqrt{\alpha}}\,\frac{d\alpha}{\alpha}.$$

The integrals along DC, FE together give

$$-(1/\pi)\int_0^\infty e^{-\alpha t}\sin a\sqrt{\alpha}(d\alpha/\alpha)$$

$$=-(2/\pi)\int_0^\infty e^{-u^2t}\sin au(du/u),$$

if we write $\alpha=u^2$. The contribution from the small circle is unity so that the inverse of $(e^{-a\sqrt{p}})/p$ is

$$1-(2/\pi)\int_0^\infty e^{-u^2t}\sin au\,(du/u).$$

The infinite integral can be expressed in terms of the error function by using the well-known result

$$\int_0^\infty e^{-\lambda^2 x^2}\cos 2\,\mu x\,dx=(\sqrt{\pi}/2\lambda)\exp(-\mu^2/\lambda^2),$$

and integrating with respect to μ from 0 to μ. The final result is

$$\operatorname{erfc}(a/2\sqrt{t}),$$

as given in the table of transforms.

2.5. AN EXAMPLE IN THE RADIAL FLOW OF HEAT

We consider here the flow of heat in a long circular cylinder of radius a and zero initial temperature when, for $t>0$, the surface is kept at a constant temperature V_0. This is one of the simplest problems on heat flow in a cylinder, but even this problem is handled rather more easily by the use of the Laplace transform than by classical methods. For problems with more complicated boundary conditions the transform method has very distinct advantages.

With the usual notation, the equations for solution are

$$\frac{\partial V^2}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial r}=\frac{1}{\kappa}\frac{\partial V}{\partial t},\qquad 0\leqslant r<a,\quad t>0,\quad .\quad (2.20)$$

with the boundary condition

$$V=V_0,\ \text{when}\ r=a,\ t>0,\ .\quad .\quad .\quad (2.21)$$

and initial condition

$$V=0,\ \text{when}\ t=0,\quad 0\leqslant r<a.\quad .\quad .\quad (2.22)$$

Writing

$$\bar{V}=\int_0^\infty e^{-pt}V\,dt,\ .\quad .\quad .\quad .\quad (2.23)$$

integrating by parts and using (2.22), we find

$$\int_0^\infty e^{-pt}(\partial V/\partial t)\,dt=p\bar{V}.\quad .\quad .\quad .\quad (2.24)$$

Multiplying (2.20), (2.21) by e^{-pt}, integrating with since $I_0'(z)=I_1(z)$ and $I_1(iz)=iJ_1(z)$. The pole at $p=0$ has residue unity, so that finally

$$V=V_0\left[1-\frac{2}{a}\sum_{n=1}^{\infty}e^{-\kappa\alpha_n^2 t}\frac{J_0(r\alpha_n)}{\alpha_n J_1(a\alpha_n)}\right]. \qquad (2.29)$$

2.6. SPECIAL SOLUTIONS FOR SMALL VALUES OF THE TIME

Solutions in the form of a series of terms containing exponentials such as we have just obtained are often unsuitable for use with small values of the time variable. For example, computation from (2.29) is very laborious if $\kappa t/a^2$ is small. Goldstein * has suggested a method of obtaining a solution well suited to computation for such values of t. The principle of the method is to expand the transform of the wanted function as an asymptotic series and then invert term by term.

If we apply this process to the example of § 2.5, since the asymptotic expansion of $I_0(z)$ is given by

$$I_0(z)\sim[e^z/\sqrt{(2\pi z)}][1+(1/8z)+(9/128z^2)+\ldots],$$

equation (2.27) gives

$$V=\sqrt{\left(\frac{a}{r}\right)}\frac{V_0}{p}\exp\left[-q(a-r)\right]\left\{1+\frac{a-r}{8qar}+\frac{9a^2-7r^2-2ar}{128q^2a^2r^2}+\ldots\right\}.$$

Using the results given in the last two entries of the table of transforms on page 20, we have

$$V=V_0\sqrt{\left(\frac{a}{r}\right)}\left[\operatorname{erfc}\left\{\frac{a-r}{2\sqrt{(\kappa t)}}\right\}+\frac{a-r}{8ar}\left(2\sqrt{(\kappa t)}\operatorname{ierfc}\left\{\frac{a-r}{2\sqrt{(\kappa t)}}\right\}\right)+\ldots\right],$$

and this is useful for small values of t, provided r/a is

* S. Goldstein, *Proc. London Math. Soc.*, *2nd Series*, Vol. 34, (1932), pp. 51–88.

not too small. Tables for ierfc x have been computed by Hartree.*

2.7. VERIFICATION OF SOLUTION

In deriving the auxiliary equations and in obtaining the wanted function from the inversion formula, assumptions as to the nature of the function to be found have to be made. Thus, in deriving the ordinary differential equation for $\bar{V}$ [equation (2.14)] in the simple heat conduction problem of § 2.3, we assumed that

$$\int_0^\infty e^{-pt}(\partial^2 V/\partial x^2)\, dt = (\partial^2/\partial x^2)\int_0^\infty e^{-pt} V\, dt.$$

Such assumptions appear reasonable in this type of problem. For complete rigour it is necessary either to appeal to a general existence theorem or to verify that the solution does in fact satisfy the original differential equation with its initial and boundary conditions. This usually has to be done for the particular problem in hand. It is a matter for consideration in any particular case whether it is best to work with the contour integral, from which the solution has been obtained by rigorous analysis, or with the final result itself.

EXAMPLES ON CHAPTER II

1. Use the Laplace transform to solve the following ordinary differential equations:

(a) $(dy/dt)+y=1$ with $y=2$ when $t=0$.

(b) $(d^2y/dt^2)+y=0$ with $y=1$, $dy/dt=0$ when $t=0$.

[(a) $y=1+e^{-t}$. (b) $y=\cos t$.]

* D. R. Hartree, *Mem. and Proc. Manchester Lit. and Phil. Soc.*, Vol. 80, (1935), p. 85. See also H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids*, Oxford, (1947), p. 373.

2. The semi-infinite solid $x<0$ has thermal diffusivity κ_1 and conductivity K_1. It is in contact along the plane $x=0$ with the semi-infinite solid $x>0$ in which these quantities are κ_2, K_2. If the initial temperature of the first solid is a constant V_0 and that of the second solid is zero, show that the temperature at time t in the second solid is

$$\frac{V_0}{1+\sigma}\ \text{erfc}\left\{\frac{x}{2\sqrt{(\kappa_2 t)}}\right\},$$

where $\sigma=(K_2\sqrt{\kappa_1}/K_1\sqrt{\kappa_2})$.

3. One face of a slab of thickness d is supplied with heat at a constant rate H and the second face is maintained at zero temperature. If κ is the diffusivity of the material of the slab, show that the ratio of the amount of heat flowing through the second face at time t to H is given by

$$1-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\exp\left\{-(n-\tfrac{1}{2})^2\frac{\pi^2\kappa t}{d^2}\right\}.$$

4. Show that an alternative solution, more suitable for small values of the time, of the problem in Ex. 3 is

$$2\sum_{n=1}^{\infty}(-1)^{n-1}\left\{1-\text{erf}\,\frac{(2n-1)d}{2\sqrt{(\kappa t)}}\right\}.$$

5. A circular membrane of unit radius is of surface density ρ and is stretched by tension T. It is at rest in its equilibrium position and at time $t=0$ a uniform pressure $\sin t$ is applied to its surface. Show that the displacement u of the membrane at radius r is given by

$$u=\frac{c^2}{T}\sin t\left\{\frac{J_0(r/c)}{J_0(1/c)}-1\right\}-\frac{2c}{T}\sum_{n=1}^{\infty}\frac{\sin c\alpha_n t\, J_0(r\alpha_n)}{\alpha_n^2(1-c^2\alpha_n^2)J_0'(\alpha_n)},$$

where $c^2=T/\rho$ and α_n is a root of $J_0(\alpha)=0$.

6. A region of unit diffusivity is bounded internally by an infinitely long cylinder of unit radius which is main-

tained at constant temperature V_0. The initial temperature of the region is zero. Show that the Laplace transform of the temperature at radius r is

$$\frac{V_0}{p}\frac{K_0(r\sqrt{p})}{K_0(\sqrt{p})}.$$

Assuming $K_0(ze^{\pm\frac{1}{2}\pi i})=\pm\frac{1}{2}\pi i[-J_0(z)\pm iY_0(z)]$, use the inversion formula to show that the temperature is given by

$$V_0+\frac{2V_0}{\pi}\int_0^\infty e^{-u^2t}\frac{J_0(ur)Y_0(u)-Y_0(ur)J_0(u)}{J_0^2(u)+Y_0^2(u)}\frac{du}{u}.$$

7. If the thermal conductivity of the region in Ex. 6 is K, show that the flux of heat at the internal surface can be written

$$KV_0\{(\pi t)^{-\frac{1}{2}}+\tfrac{1}{2}-\tfrac{1}{4}(t/\pi)^{\frac{1}{2}}+\ldots\},$$

a form suitable for use with small values of t.

CHAPTER III

FOURIER TRANSFORMS

3·1. Fourier transforms can be used in a similar way to reduce the number of independent variables in a partial differential equation. We first discuss the use of the sine and cosine transforms and in the later sections of the present chapter examples are given in which the complex Fourier transform has proved useful.

3·2. SINE AND COSINE TRANSFORMS

These transforms can be employed when the range of the variable selected for exclusion is 0 to ∞. Care, of course, has to be taken that the integrals defining the transforms exist—the exponential term in the integral of the Laplace transform is now replaced by a sine or cosine and the necessary conditions for existence are more stringent.

The choice of sine or cosine transform is decided by the form of the boundary conditions at the lower limit of the variable selected for exclusion. Suppose V is the wanted function and we are removing a term $\partial^2 V/\partial x^2$ from the differential equation. If a sine transform is being used, we multiply the differential equation by $\sin px$ and integrate with respect to x from 0 to ∞. Integration by parts gives

$$\int_0^\infty \sin px(\partial^2 V/\partial x^2)\,dx = \Big[(\partial V/\partial x)\sin px\Big]_{x=0}^{\infty} - p\int_0^\infty \cos px\,(\partial V/\partial x)\,dx.$$

The first term of the right-hand side vanishes at the lower limit through the sine term. It vanishes also at the upper limit if V is such that $\partial V/\partial x$ tends to zero as x

tends to infinity: this is usually the case in physical problems. If this is so, a second integration gives

$$\int_0^\infty \sin px(\partial^2 V/\partial x^2)\,dx = -p\Big[V\cos px\Big]_{x=0}^\infty - p^2\int_0^\infty V\sin px\,dx.$$

Assuming for the moment that V tends to zero as x tends to infinity, this gives

$$\int_0^\infty \sin px(\partial^2 V/\partial x^2)\,dx = p[V]_{x=0} - p^2\bar{V}, \qquad (3.1)$$

where $[V]_{x=0}$ is the value of V when $x=0$ and $\bar{V}$ is the sine transform of V.

Similarly two integrations by parts give

$$\int_0^\infty \cos px(\partial^2 V/\partial x^2)\,dx = -[\partial V/\partial x]_{x=0} - p^2\bar{V}, \qquad (3.2)$$

where $\bar{V}$ is now the cosine transform of V and we have made the same assumptions regarding the behaviour of V and $\partial V/\partial x$ as x tends to infinity.

Thus the successful use of a sine transform in removing a term $\partial^2 V/\partial x^2$ from a differential equation requires a knowledge of the value of V when $x=0$, while to use a cosine transform for this purpose we need to know $\partial V/\partial x$ when $x=0$. Similar but, of course, more extensive considerations hold when we attempt to remove a term like $\partial^4 V/\partial x^4$ (or any other derivative of even order).

It should be noted that whereas terms $\partial V/\partial x$ or $\partial^2 V/\partial x^2$ can be removed by a Laplace transform, a term $\partial V/\partial x$ (or any derivative of odd order) cannot be removed by a sine or cosine transform for a single integration by parts leaves $\int_0^\infty V \begin{matrix}\cos\\ \sin\end{matrix}\, px\,dx$ in the expression for $\int_0^\infty \left(\frac{\partial V}{\partial x}\right) \begin{matrix}\sin\\ \cos\end{matrix}\, px\,dx$. However, if a sine or cosine transform can be used in the solution of a given differential equation a considerable advantage is often obtained over the Laplace transform. The inversion is very much easier in that a real integral takes the place of a contour integral in the inversion formula.

3·3. THE HEAT CONDUCTION PROBLEM OF § 2.3 SOLVED BY A SINE TRANSFORM

As a simple example, the sine transform is here employed to find the temperature in a semi-infinite solid with zero initial and constant surface temperature. This is the problem solved in § 2.3 by the Laplace transform and the equations for solution are (2.9), (2.10) and (2.11). Since the temperature V is given ($=V_0$, constant) when $x=0$, the sine transform,

$$\bar{V}=\int_0^\infty V \sin px\, dx, \quad . \quad . \quad . \quad . \quad (3\cdot3)$$

is appropriate.

We therefore multiply the differential equation (2.9) and initial condition (2.11) by the kernel $\sin px$ and integrate with respect to x between 0 and infinity. On physical grounds it is clear that both V and $\partial V/\partial x$ tend to zero as x tends to infinity and we can use (3.1) with $[V]_{x=0}=V_0$. The auxiliary equations are therefore

$$d\bar{V}/dt=\kappa(pV_0-p^2\bar{V}), \qquad t>0, \quad . \quad . \quad (3\cdot4)$$

and

$$\bar{V}=0, \text{ when } t=0. \quad . \quad . \quad . \quad . \quad (3\cdot5)$$

Once again we have reduced the problem to the solution of an ordinary differential equation.

The solution of (3·4), finite for $t>0$ and satisfying (3·5), is easily seen to be

$$\bar{V}=(V_0/p)[1-e^{-p^2\kappa t}]. \quad . \quad . \quad . \quad . \quad (3\cdot6)$$

The inversion formula (1.12) gives

$$V=(2V_0/\pi)\int_0^\infty [1-e^{-p^2\kappa t}] \sin xp\, (dp/p).$$

Since

$$\int_0^\infty \sin xp\,(dp/p)=\pi/2,$$

this can be written

$$V=V_0[1-(2/\pi)\int_0^\infty e^{-p^2\kappa t}\sin xp\,(dp/p)]$$
$$=V_0\,\mathrm{erfc}\{x/2\sqrt{(\kappa t)}\},$$

as shown on page 26. It is at once apparent that the inversion to V is much easier with a sine transform.

3·4. AN EXAMPLE OF THE REPEATED USE OF TRANSFORMS

Examples have already been given in which the application of an integral transform reduces a problem in two independent variables to one involving an ordinary differential equation. If we start with a problem involving three independent variables, two transforms applied successively will reduce the governing equation to an ordinary differential equation. This process can be repeated and in theory be applied to an equation in n variables, but, of course, the final result will be complicated by the successive inversions which have to be applied to obtain the final solution. Details of the process are shown in the example given below.

Suppose we wish to find the temperature V in a long cylinder of radius a and diffusivity κ with initial temperature zero, when for $t>0$ the surface is kept at unit temperature over a band of width $2c$ in the middle of the cylinder and at zero outside this band.

Here we have to find V from

$$\frac{\partial^2 V}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial r}+\frac{\partial^2 V}{\partial z^2}=\frac{1}{\kappa}\frac{\partial V}{\partial t},$$

$$0\leqslant r<a, \qquad -\infty<z<\infty, \qquad t>0, \quad . \quad (3{\cdot}7)$$

with the boundary condition

$$\left.\begin{array}{ll} V=1, & |z|<c, \\ V=0, & |z|>c, \end{array}\right\} \text{ when } r=a,\ t>0, \quad . \quad (3.8)$$

and initial condition

$$V=0, \text{ when } t=0, \quad 0\leqslant r<a, \quad -\infty<z<\infty. \quad (3.9)$$

This then is a problem in which the governing equation contains three independent variables r, z and t.

The temperature distribution is clearly symmetrical about the plane $z=0$ and hence $\partial V/\partial z=0$ there. Thus we can use a cosine transform and work with the range 0, ∞ in z. If therefore we write

$$\bar{V}_c=\int_0^\infty V \cos pz\, dz, \quad . \quad . \quad . \quad (3.10)$$

since it is clear physically that V and $\partial V/\partial z$ will tend to zero as z tends to infinity, equation (3.2) gives

$$\int_0^\infty \cos pz\, (\partial^2 V/\partial z^2)\, dz=-p^2\bar{V}_c. \quad . \quad (3.11)$$

Multiplying equations (3.7), (3.8) and (3.9) by the kernel $\cos pz$, integrating with respect to z between 0 and ∞ and making use of (3.10), (3.11), we have as auxiliary equations for $\bar{V}_c$ in the two independent variables r and t

$$\frac{\partial^2\bar{V}_c}{\partial r^2}+\frac{1}{r}\frac{\partial\bar{V}_c}{\partial r}-p^2\bar{V}_c=\frac{1}{\kappa}\frac{\partial\bar{V}_c}{\partial t}, \quad 0\leqslant r<a, \quad t>0, \quad . \quad (3.12)$$

$$\bar{V}_c=\int_0^c \cos pz\, dz=(\sin pc)/p, \text{ when } r=a, \quad t>0, \quad . \quad (3.13)$$

$$\bar{V}_c=0, \text{ when } t=0, \quad 0\leqslant r<a. \quad . \quad . \quad (3.14)$$

The solution of the boundary-value problem specified by equations (3.12), (3.13) and (3.14) can be obtained by the application of the Laplace transform. We write

$$\bar{V}_L=\int_0^\infty e^{-p't}\bar{V}_c\, dt. \quad . \quad . \quad . \quad . \quad (3.15)$$

Integration by parts and use of (3.14) gives

$$\int_0^\infty e^{-p't}(\partial \bar{V}_c/\partial t)\,dt = p'\bar{V}_L. \quad . \quad . \quad (3.16)$$

Multiplying (3.12) and (3.13) by $e^{-p't}$, integrating with respect to t between 0 and ∞, and using (3.15), (3.16) leaves us with the ordinary differential equation

$$\frac{d^2\bar{V}_L}{dr^2}+\frac{1}{r}\frac{d\bar{V}_L}{dr}-(p^2+p'/\kappa)\bar{V}_L=0, \quad 0\leqslant r<a, \quad . \quad (3.17)$$

and the boundary condition

$$\bar{V}_L=\{(\sin pc)/p\}\int_0^\infty e^{-p't}\,dt$$

$$=(\sin pc)/(pp'), \text{ when } r=a. \quad . \quad . \quad (3.18),$$

The solution of (3.17), finite at $r=0$, and satisfying (3.18) is

$$\bar{V}_L=\frac{\sin pc}{pp'}\frac{I_0[\sqrt{(p^2+p'/\kappa)}r]}{I_0[\sqrt{(p^2+p'/\kappa)}a]}. \quad . \quad . \quad (3.19)$$

Inversion to $\bar{V}_c$ is performed by the inversion formula (1.10) for the Laplace transform, giving

$$\bar{V}_c=\frac{1}{2\pi i}\frac{\sin pc}{p}\int_{\gamma-i\infty}^{\gamma+i\infty}e^{p't}\frac{I_0[\sqrt{(p^2+p'/\kappa)}r]}{I_0[\sqrt{(p^2+p'/\kappa)}a]}\frac{dp'}{p'}. \quad . \quad (3.20)$$

Using the contour of Fig. 1 and the usual procedure, the line integral can be replaced by $2\pi i$ times the sum of the residues at the poles of the integrand within the contour. These poles are at $p'=0$ and $p'=-\kappa(\alpha_s^2+p^2)$ where α_s is a root of $J_0(\alpha_s a)=0$. The pole at $p'=0$ has residue

$$I_0(pr)/I_0(pa),$$

while the other poles give residues

$$\frac{\exp[-\kappa(\alpha_s^2+p^2)t]I_0(i\alpha_s r)}{\left[p'\dfrac{d}{dp'}\{I_0[\sqrt{(p^2+p'/\kappa)}a]\}\right]_{p'=-\kappa(\alpha_s^2+p^2)}}.$$

The denominator

$$=(1/2\kappa)\{ap'(p^2+p'/\kappa)^{-\frac{1}{2}}I_0'[\sqrt{(p^2+p'/\kappa)}a]\}_{p'=-\kappa(\alpha_s^2+p^2)}$$
$$=-(a/2\alpha_s)(\alpha_s^2+p^2)J_1(\alpha_s a),$$

for $I_1(iz)=iJ_1(z)$. Hence

$$\bar{V}_c=\frac{\sin pc}{p}\left[\frac{I_0(pr)}{I_0(pa)}-\frac{2}{a}\sum_{s=1}^{\infty}\frac{\alpha_s}{\alpha_s^2+p^2}\frac{J_0(\alpha_s r)}{J_1(\alpha_s a)}\exp\{-\kappa(\alpha_s^2+p^2)t\}\right]. \quad (3.21)$$

Inversion to V is given by the formula (1.13) for the cosine transform, viz.,

$$V=(2/\pi)\int_0^\infty \bar{V}_c \cos zp\, dp,$$

for now z is the variable involved. Thus we arrive at the final result

$$V=\frac{2}{\pi}\int_0^\infty \frac{I_0(pr)}{I_0(pa)}\frac{\sin cp}{p}\cos zp\, dp$$
$$-\frac{4}{\pi a}\sum_{s=1}^{\infty}e^{-\kappa\alpha_s^2 t}\frac{\alpha_s J_0(\alpha_s r)}{J_1(\alpha_s a)}\int_0^\infty e^{-\kappa t p^2}\frac{\sin cp\cos zp}{\alpha_s^2+p^2}\frac{dp}{p}. \quad (3.22)$$

3.5. THE COMPLEX FOURIER TRANSFORM

When one of the variables ranges from $-\infty$ to $+\infty$, it can often be excluded by the use of the complex Fourier transform. We give first an example in which the transform defined in equation (1.4) and its inversion formula (1.15) can be used.

3.6. THE MOTION OF A VERY LONG STRING

We consider the motion of a very long string, fixed at its two ends, of mass ρ per unit length and under tension P, when it is displaced into the curve $y=f(x)$ and let go from rest. If y is the displacement at time t, we have to solve

$$c^2(\partial^2 y/\partial x^2)=\partial^2 y/\partial t^2, \quad -\infty<x<\infty, \quad t>0, \quad (3.23)$$

where

$$c^2=P/\rho,$$

subject to the initial conditions

$$\left.\begin{matrix} y=f(x), \\ \partial y/\partial t=0, \end{matrix}\right\} \text{ when } t=0, \quad -\infty<x<\infty. \quad (3.24)$$

We write

$$\bar{y}=\int_{-\infty}^{\infty} ye^{ipx}\,dx; \quad (3.25)$$

integrating by parts and noting that both y and $\partial y/\partial x$ are zero at the ends of the string, we find

$$\int_{-\infty}^{\infty} e^{ipx}(\partial^2 y/\partial x^2)\,dx=-p^2\bar{y}. \quad (3.26)$$

Multiplying equation (3.23) and the initial conditions (3.24) by the kernel e^{ipx} of the Fourier transform (3.25), integrating with respect to x between $-\infty$ and ∞ and using (3.26), we have as the auxiliary equations

$$d^2\bar{y}/dt^2=-c^2p^2\bar{y}, \quad t>0, \quad (3.27)$$

with

$$\left.\begin{matrix} \bar{y}=\bar{f}(p)=\displaystyle\int_{-\infty}^{\infty} f(x')e^{ipx'}\,dx', \\ d\bar{y}/dt=0, \end{matrix}\right\} \text{ when } t=0. \quad (3.28)$$

The dashes have been introduced in the integral of the first of equations (3.28) to avoid later confusion with x.

The solution of the auxiliary equations is

$$\bar{y}=\bar{f}(p)\cos cpt,$$

and the inversion formula (1.15) gives for y

$$2\pi y=\int_{-\infty}^{\infty} \bar{f}(p)\cos ctpe^{-ixp}\,dp. \quad (3.29)$$

Using the expression for $\bar{f}(p)$ given in the first of (3.28), this can be written, when we replace the cosine by its exponential value and write $p=-\alpha$,

$$4\pi y=\int_{-\infty}^{\infty}\left[e^{(x-ct)i\alpha}+e^{(x+ct)i\alpha}\right]d\alpha\int_{-\infty}^{\infty}f(x')e^{-i\alpha x'}\,dx'.$$

If we now use Fourier's integral formula in the form (1.8), we have

$$y=\tfrac{1}{2}[f(x-ct)+f(x+ct)],$$

which is the classical form of the solution to this problem.

3.7. STRESSES IN A LONG CIRCULAR CYLINDER

As an application of the complex Fourier transform in the more general form given by equations (1.16), we consider the stress distribution in a long circular cylinder when a discontinuous pressure is applied to the curved surface. The axis of the cylinder is taken as the z-axis, the origin being at the central cross-section, and the usual cylindrical coordinates r, θ, z are employed. The radius of the cylinder is taken as unity* and, for $z>0$, unit pressure is applied to the surface $r=1$ while, for $z<0$, the boundary is unloaded. The shear stress over the curved surface is taken as zero for the whole length of the cylinder. With trifling modifications the method can be used for applied stresses under fairly wide restrictions, e.g., that they be of exponential type.

For axially-symmetrical stresses in a cylinder, the radial, shear, hoop and longitudinal stresses are given in terms of a stress function χ by the equations†

$$\widehat{rr}=(\partial/\partial z)\{\sigma\nabla^2-(\partial^2/\partial r^2)\}\chi, \quad (3.30)$$

$$\widehat{rz}=(\partial/\partial r)\{(1-\sigma)\nabla^2-(\partial^2/\partial z^2)\}\chi, \quad (3.31)$$

$$\widehat{\theta\theta}=(\partial/\partial z)\{\sigma\nabla^2-r^{-1}(\partial/\partial r)\}\chi, \quad (3.32)$$

$$\widehat{zz}=(\partial/\partial z)\{(2-\sigma)\nabla^2-(\partial^2/\partial z^2)\}\chi, \quad (3.33)$$

* The solution for a cylinder of radius a can be obtained by writing r/a, z/a for r, z respectively.

† A. E. H. Love, *Mathematical Theory of Elasticity*, Cambridge, 3rd Ed., (1920), p. 278.

where σ is Poisson's ratio of the material of the cylinder, ∇^2 denotes the Laplace operator

$$(\partial^2/\partial r^2)+r^{-1}(\partial/\partial r)+(\partial^2/\partial z^2)$$

and χ satisfies the biharmonic equation

$$\nabla^4\chi=0, \quad -\infty<z<\infty, \quad r<1. \quad . \ (3.34)$$

Equation (3.34) is the governing differential equation for this problem, the boundary conditions being

$$\left.\begin{aligned} \widehat{rr}&=-1,\ 0<z<\infty \\ &=0,\ -\infty<z<0 \end{aligned}\right\}, \text{ when } r=1, \quad . \ (3.35)$$

$$\widehat{rz}=0, \text{ when } r=1, \quad -\infty<z<\infty, \quad . \ (3.36)$$

the expressions for $\widehat{rr}$, $\widehat{rz}$ in terms of the stress function χ being given by (3.30) and (3.31).

In the problem of the last section (§ 3.6), the wanted function vanished at both ends of the doubly infinite range of the variable excluded. Here the wanted function χ does not vanish when z tends to plus infinity and we have to use the more general transform given by (1.16). We write

$$\begin{aligned}\bar{\chi}&=\bar{\chi}_++\bar{\chi}_- \\ &=\int_0^\infty \chi e^{ipz}\,dz+\int_{-\infty}^0 \chi e^{ipz}\,dz, \quad . \quad . \ (3.37)\end{aligned}$$

where, to secure the existence of the integrals, the imaginary part of p has to be chosen appropriately. We assume that

$$\begin{aligned}\chi&=A+O(e^{-cz}) \text{ as } z\rightarrow+\infty, \\ &=O(e^{cz}) \text{ as } z\rightarrow-\infty,\end{aligned}$$

where A and c are suitable constants. It is then sufficient to take the imaginary part of p to be k where $k<c$.

Integration by parts gives

$$\int_{-\infty}^{\infty} e^{ipz}(\partial^n\chi/\partial z^n)\,dz=(-ip)^n\,\bar{\chi},$$

for $n=1, 2, 3, 4$. Multiplying equation (3.34) by the kernel

e^{ipz} and integrating with respect to z from $-\infty$ to ∞ gives, if we denote $(d^2/dr^2)+r^{-1}(d/dr)-p^2$ by $\bar{\nabla}^2$, the ordinary fourth order differential equation

$$\bar{\nabla}^4\bar{\chi}=0, \qquad r<1. \quad . \quad . \quad . \quad . \quad (3\cdot38)$$

A similar operation on the boundary conditions (3·35), (3·36) leads to

$$\{\sigma\bar{\nabla}^2-(d^2/dr^2)\}\bar{\chi}=p^{-2}, \text{ when } r=1, \quad . \quad (3\cdot39)$$

and

$$(d/dr)\{(1-\sigma)\bar{\nabla}^2+p^2\}\bar{\chi}=0, \text{ when } r=1. \quad (3\cdot40)$$

Equation (3·38) with the boundary conditions (3·39), (3·40) are the auxiliary equations for this problem. The solution of (3·38), finite at $r=0$, is

$$\bar{\chi}=AI_0(pr)+BprI_1(pr), \quad . \quad . \quad . \quad (3\cdot41)$$

where A and B are constants. Using the recurrence formulae for the Bessel functions,* we find

$$\bar{\nabla}^2\bar{\chi}=2Bp^2I_0(pr), \quad . \quad . \quad . \quad . \quad (3\cdot42)$$

and substitution in (3·39) and (3·40) yields

$$A\{I_0(p)-p^{-1}I_1(p)\}+B\{(1-2\sigma)I_0(p)+pI_1(p)\}=-p^{-4}, \quad (3\cdot43)$$

$$AI_1(p)+B\{2(1-\sigma)I_1(p)+pI_0(p)\}=0. \quad . \quad (3\cdot44)$$

Hence

$$\left.\begin{aligned} p^3D(p)A&=2(1-\sigma)I_1(p)+pI_0(p),\\ p^3D(p)B&=-I_1(p), \end{aligned}\right\} \quad . \quad . \quad (3\cdot45)$$

where

$$D(p)=\{p^2+2(1-\sigma)\}I_1^2(p)-p^2I_0^2(p). \quad . \quad (3\cdot46)$$

Substitution in (3·41) then gives $\bar{\chi}$, the transform of the stress function, completely.

We are interested in the stresses themselves rather than in the stress function χ. Denoting the transform of a stress by a "bar", so that

$$\overline{\theta\theta}=\int_{-\infty}^{\infty}\widehat{\theta\theta}e^{ipz}\,dz, \text{ etc.,}$$

* G. N. Watson, *Bessel Functions*, Cambridge, (1944), § 3·71.

multiplication of equation (3.32) by e^{ipz}, integration with respect to z from $-\infty$ to ∞ gives

$$\overline{\theta\theta} = -ip\{\sigma\overline{\nabla}^2 - r^{-1}(d/dr)\}\bar{\chi}.$$

The transforms of the other stresses can be found similarly. Using (3.41), (3.42) and (3.45), we find

$$\overline{\theta\theta} = \{ip^{-1}r^{-1}D^{-1}(p)\}\{pI_0(p)I_1(pr) + 2(1-\sigma)I_1(p)I_1(pr) + (2\sigma-1)prI_1(p)I_0(pr)\}, \quad (3.47)$$

with similar expressions for the transforms of the other stresses.

The inversion formula for (3.37) is given by (1.18) with the imaginary parts of the limits ik in all cases. We can therefore write

$$2\pi\widehat{\theta\theta} = \int_{ik-\infty}^{ik+\infty} \overline{\theta\theta} e^{-izp}\, dp,$$

with similar expressions for the other stresses. In each case the integrand has no poles on the real axis, apart from the origin. The contour of the integral may therefore be deformed into the real axis from $-\infty$ to $+\infty$ with a clockwise semi-circle round the origin. The integral may then be evaluated as the sum of the integrals along the real axis less πi times the residue of the integrand at $p=0$. The residues at $p=0$ for the various stresses can be found by using the power series for the Bessel functions. In the case of the hoop stress $\widehat{\theta\theta}$, the residue is $-i$. When we combine the integrals along the negative and positive parts of the real axis, we find

$$\widehat{\theta\theta} = -(1/2) + (\pi r)^{-1}\int_0^\infty T(p)p^{-1}\sin zp\, dp, \quad (3.48)$$

where

$$T(p) = D^{-1}(p)\{pI_0(p)I_1(rp) + 2(1-\sigma)I_1(p)I_1(rp) + (2\sigma-1)rpI_1(p)I_0(rp)\}, \quad (3.49)$$

and $D(p)$ has been defined in (3.46).

For details of the formulae giving the other stresses, for expressions giving the displacements and for numerical results, the reader is referred to the original paper dealing with this problem.* An outline of a suggested method for the numerical calculation of integrals such as appear in (3.48) is given in Chapter V.

EXAMPLES ON CHAPTER III

1. Use a cosine transform to show that the steady temperature in the semi-infinite solid $y>0$, when the temperature on the surface $y=0$ is kept at unity over the strip $|x|<a$ and at zero outside this strip, is

$$\frac{1}{\pi}\left[\tan^{-1}\left(\frac{a+x}{y}\right)+\tan^{-1}\left(\frac{a-x}{y}\right)\right].$$

The result $\int_0^\infty e^{-sx}x^{-1}\sin rx\,dx=\tan^{-1}(r/s)$, $r>0$, $s>0$, may be assumed.

2. Show that the solution of Laplace's equation for V inside the semi-infinite strip $x>0$, $0<y<b$, such that

$$\begin{aligned} V&=f(x), &&\text{when } y=0, && 0<x<\infty,\\ V&=0, &&\text{when } y=b, && 0<x<\infty,\\ V&=0, &&\text{when } x=0, && 0<y<b, \end{aligned}$$

is given by

$$V=\frac{2}{\pi}\int_0^\infty f(u)\,du\int_0^\infty\frac{\sinh (b-y)p}{\sinh bp}\sin xp\sin up\,dp.$$

3. Show how the solution of the problem of heat conduction in a semi-infinite solid with zero initial and constant surface temperature can, by the use of a sine transform followed by a Laplace transform, be made

* C. J. Tranter and J. W. Craggs, *Phil. Mag.*, 7, XXXVI, (1945), pp. 241–250.

to depend on the solution of an algebraic equation. Complete the solution of the heat conduction problem by this method.

4. Use a sine transform to show that the steady temperature in the semi-infinite annulus bounded by the plane $z=0$ and the cylindrical surfaces $r=a$, $r=b$, $(a>b)$, when the surfaces $z=0$, $r=a$ are maintained at zero and the surface $r=b$ is kept at temperature $f(z)$, is

$$(2/\pi)\int_0^\infty\int_0^\infty F(a, b, r, p)f(z') \sin pz \sin pz' \, dp \, dz',$$

where

$$F(a, b, r, p)=\frac{I_0(pr)K_0(pa)-I_0(pa)K_0(pr)}{I_0(pb)K_0(pa)-I_0(pa)K_0(pb)}.$$

5. Use the complex form of the Fourier transform to show that

$$V=\frac{1}{2\sqrt{(\pi t)}}\int_{-\infty}^\infty f(u) \exp\left\{\frac{-(x-u)^2}{4t}\right\} du$$

is the solution of the boundary-value problem

$$\partial V/\partial t=\partial^2 V/\partial x^2, \qquad -\infty<x<\infty, \qquad t>0,$$
$$V=f(x), \text{ when } t=0.$$

6. Show that the shear stress in the problem of § 3.7 is given by

$$\widehat{rz}=(1/\pi)\int_0^\infty S(p) \cos zp \, dp$$

where

$$S(p)=D^{-1}(p)\{pI_1(rp)I_0(p)-rpI_1(p)I_0(rp)\},$$

and $D(p)$ is given by the equation (3.46).

CHAPTER IV

HANKEL AND MELLIN TRANSFORMS

4·1. For problems in which there is symmetry about an axis, polar coordinates are appropriate. If the range of the radial variable is 0 to ∞, it can be removed conveniently by the application of a Hankel transform. Two examples of the procedure, which is essentially similar to that for the Laplace and Fourier transforms, are given below. In the first (§ 4.2), the problem is governed by the biharmonic equation and the boundary conditions are comparatively simple to handle since the same functions of the wanted function are given over the whole boundary. In the second (§ 4·3), the governing equation is a simpler one but the boundary condition is much more difficult to handle in that different functions of the quantity sought are prescribed over different parts of the same boundary.

4·2. THE PROBLEM OF BOUSSINESQ

The stress distribution in a semi-infinite solid due to a load acting over part of its boundary has been given by Boussinesq in a series of papers in Comptes Rendus (1878–1883). Love* has given an alternative solution for the case where the load acts inside a circular boundary. As an example of the use of the Hankel transform, the following special case is here considered. A uniform pressure of unit intensity is applied over a circle of unit radius on the surface of the semi-infinite solid $z>0$, the rest of the surface $z=0$ being stress-free. The magnitude of the stress in a direction perpendicular to the surface of the solid at a point on the axis of the pressed area distant z from the surface is required.

* A. E. H. Love, *Phil. Trans. Roy. Soc., Series A*, 228, (1929), pp. 377–420.

As in § 3.7, the normal and shear stresses are given in terms of a stress function χ by

$$\widehat{zz}=(\partial/\partial z)\{(2-\sigma)\nabla^2-(\partial^2/\partial z^2)\}\chi, \quad . \quad (4.1)$$

$$\widehat{rz}=(\partial/\partial r)\{(1-\sigma)\nabla^2-(\partial^2/\partial z^2)\}\chi, \quad . \quad (4.2)$$

where σ is Poisson's ratio of the solid, ∇^2 denotes the Laplace operator $(\partial^2/\partial r^2)+r^{-1}(\partial/\partial r)+(\partial^2/\partial z^2)$ and χ satisfies the biharmonic equation

$$\nabla^4\chi=0, \qquad 0<r<\infty, \qquad z>0. \quad . \quad (4.3)$$

The boundary conditions are

$$\left.\begin{aligned}\widehat{zz}&=-1,\ 0<r<1\\ &=0,\ r>1\end{aligned}\right\}, \text{ when } z=0, \quad . \quad (4.4)$$

$$\widehat{rz}=0, \text{ when } z=0, \qquad 0<r<\infty. \quad . \quad (4.5)$$

Before proceeding to the standard method of applying an integral transform, it is useful to obtain the definite integral

$$I=\int_0^\infty \nabla^2 frJ_0(pr)\,dr, \quad . \quad . \quad . \quad (4.6)$$

where f is a function of r and z. We have

$$I=\int_0^\infty r\left(\frac{\partial^2 f}{\partial r^2}+\frac{1}{r}\frac{\partial f}{\partial r}\right)J_0(pr)\,dr+\frac{\partial^2}{\partial z^2}\int_0^\infty frJ_0(pr)\,dr, \quad (4.7)$$

and the first integral can be written

$$\int_0^\infty \frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right)J_0(pr)\,dr=\left[r\frac{\partial f}{\partial r}J_0(pr)\right]_0^\infty-p\int_0^\infty\frac{\partial f}{\partial r}rJ_0'(pr)\,dr.$$

We assume f to be such that the first term on the right-hand side vanishes at both limits, and a second integration by parts then gives for the integral under discussion

$$-p\Big[frJ_0'(pr)\Big]_0^\infty+p\int_0^\infty f\{J_0'(pr)+prJ_0''(pr)\}\,dr.$$

Assuming further that f is such that the first term vanishes at both limits and noting that $J_0(pr)$ satisfies the equation

$$J_0''(pr)+(pr)^{-1}J_0'(pr)+J_0(pr)=0,$$

substitution in (4.7) gives

$$I=[(\partial^2/\partial z^2)-p^2]\int_0^\infty frJ_0(pr)\,dr. \quad . \quad . \quad (4.8)$$

If we write $f=\nabla^2\chi$, repeated applications of (4.6) and (4.8) give

$$\int_0^\infty \nabla^4\chi rJ_0(pr)\,dr=[(\partial^2/\partial z^2)-p^2]\int_0^\infty \nabla^2\chi rJ_0(pr)\,dr$$

$$=[(\partial^2/\partial z^2)-p^2]^2\int_0^\infty \chi rJ_0(pr)\,dr. \quad (4.9)$$

As usual, we denote the transform of a quantity by a "bar" so that

$$\bar{\chi}=\int_0^\infty \chi rJ_0(pr)\,dr, \quad . \quad . \quad . \quad (4.10)$$

$$\overline{\widehat{zz}}=\int_0^\infty \widehat{zz}rJ_0(pr)\,dr, \quad . \quad . \quad . \quad (4.11)$$

and equation (4.9) can be written

$$\int_0^\infty \nabla^4\chi rJ_0(pr)\,dr=[(\partial^2/\partial z^2)-p^2]^2\bar{\chi}. \quad . \quad (4.12)$$

If we write $f=\chi$ in (4.8), equation (4.1) gives, after slight reduction,

$$\overline{\widehat{zz}}=(1-\sigma)(\partial^3\bar{\chi}/\partial z^3)-(2-\sigma)p^2(\partial\bar{\chi}/\partial z). \quad . \quad (4.13)$$

Thus, by multiplying by $rJ_0(pr)$ and integrating with respect to r from 0 to ∞, the governing equation (4.3) and the first boundary condition (4.4) become, since $\bar{\chi}$ is a function of z (and p) only,

$$[(d^2/dz^2)-p^2]^2\bar{\chi}=0, \quad . \quad . \quad . \quad (4.14)$$

and, when $z=0$,

$$(1-\sigma)(d^3\bar{\chi}/dz^3)-(2-\sigma)p^2(d/\bar{\chi}dz)$$

$$=-\int_0^1 rJ_0(pr)\,dr=-p^{-1}J_1(p). \quad . \quad (4.15)$$

A slightly different procedure is adopted for the transformation of the second boundary condition (4.5). If g is a function of r and z, integration by parts shows that, since $J_1(pr) = -J_0'(pr)$,

$$\int_0^\infty \frac{\partial g}{\partial r} r J_1(pr)\, dr = \Big[g r J_1(pr) \Big]_0^\infty + \int_0^\infty g \{ J_0'(pr) + pr J_0''(pr) \}\, dr.$$

If we assume that g is such that the first term on the right-hand side vanishes at both limits and make use of the differential equation satisfied by $J_0(pr)$, this can be written

$$\int_0^\infty \frac{\partial g}{\partial r} r J_1(pr)\, dr = -p \int_0^\infty g r J_0(pr)\, dr. \quad . \quad (4.16)$$

Writing

$$g = \{(1-\sigma)\nabla^2 - (\partial^2/\partial z^2)\}\chi,$$

equations (4.2) and (4.5) therefore give, when $z=0$,

$$\int_0^\infty \{(1-\sigma)\nabla^2 - (\partial^2/\partial z^2)\}\chi r J_0(pr)\, dr = 0.$$

Use of (4.8) with $f=\chi$ then gives as the transform of the second boundary condition

$$\sigma(d^2\bar{\chi}/dz^2) + (1-\sigma)p^2\bar{\chi} = 0, \text{ when } z=0. \quad . \quad (4.17)$$

The problem is therefore reduced to the solution of the fourth order ordinary differential equation (4.14) subject to the two conditions (4.15) and (4.17). The general solution of (4.14), finite for large positive z, is

$$\bar{\chi} = (A+Bz)e^{-pz}, \quad . \quad . \quad . \quad . \quad (4.18)$$

giving

$$d\bar{\chi}/dz = [B(1-pz) - Ap]e^{-pz},$$
$$d^2\bar{\chi}/dz^2 = [B(-2p+p^2z) + Ap^2]e^{-pz},$$
$$d^3\bar{\chi}/dz^3 = [B(3p^2-p^3z) - Ap^3]e^{-pz}.$$

Writing $z=0$ and substituting in (4.15), (4.17), we therefore have

$$pA + (1-2\sigma)B = -p^{-3}J_1(p),$$
$$pA - 2\sigma B = 0,$$

as the equations determining A and B.

These give

$$A = -2\sigma p^{-1} J_1(p), \qquad B = -p^{-3} J_1(p). \qquad . \quad (4.19)$$

Substitution from (4.19), (4.18) in (4.13) and some reduction then gives for the transform of the $\widehat{zz}$ stress,

$$\overline{zz} = -(z+p^{-1}) J_1(p) e^{-pz}.$$

From the inversion formula (1.24) for the Hankel transform

$$\widehat{zz} = \int_0^\infty \overline{zz}\, p J_0(rp)\, dp,$$

we have

$$\widehat{zz} = -\int_0^\infty (1+pz) e^{-pz} J_1(p) J_0(rp)\, dp. \qquad . \quad (4.20)$$

On the axis, $r=0$, and this reduces to

$$\widehat{zz} = -\int_0^\infty (1+pz) e^{-pz} J_1(p)\, dp$$
$$= -1 + z^3(z^2+1)^{-3/2},$$

when use is made of the results given by Watson.*

The other stresses, if required, can be obtained by finding the transforms $\overline{rr}$, $\overline{\theta\theta}$ by using equations (3.30), (3.32) and setting $f=\chi$ in (4.8). Inversion by (1.24) with kernel $rJ_0(pr)$ then gives the stresses themselves. The transform of the shear stress $\widehat{rz}$ is found by writing

$$g = \{(1-\sigma)\nabla^2 - (\partial^2/\partial z^2)\}\chi$$

in (4.16) and using $f=\chi$ in (4.8). This gives the Hankel transform with kernel $rJ_1(pr)$ and inversion in this case is performed by using (1.24) with this kernel.

4.3. THE ELECTRIFIED DISC

As a second example of the use of the Hankel transform, we give a solution on these lines to Weber's classi-

* G. N. Watson, *Bessel Functions*, Cambridge, (1944), p. 386.

cal problem of the field due to an electrified disc. Let V be the potential due to a flat circular disc, radius unity, the centre of the disc being at the origin and its axis along the z-axis. In polar coordinates the potential satisfies Laplace's equation

$$\nabla^2 V \equiv \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial z^2} = 0, \quad . \quad . \quad . \quad (4.21)$$

with the boundary conditions

$$\left.\begin{array}{ll} V = V_0, & 0 \leqslant r < 1, \\ \partial V/\partial z = 0, & r > 1, \end{array}\right\} \text{ when } z = 0, \quad . \quad (4.22)$$

where V_0 is the assumed (constant) potential of the disc and the second of (4.22) arises from the symmetry about the plane $z=0$.

We write

$$\bar{V} = \int_0^\infty V r J_0(pr)\, dr, \quad . \quad . \quad . \quad . \quad (4.23)$$

and, as in the derivation of (4.8), find

$$\int_0^\infty \nabla^2 V r J_0(pr)\, dr = [(\partial^2/\partial z^2) - p^2]\bar{V}.$$

Multiplication of (4.21) by $rJ_0(pr)$ and integration with respect to r from 0 to ∞ therefore yields

$$d^2\bar{V}/dz^2 = p^2\bar{V}. \quad . \quad . \quad . \quad . \quad . \quad (4.24)$$

The boundary condition (4.22) is a "mixed" one, in that V is specified from $0 \leqslant r < 1$ and $\partial V/\partial z$ for $r > 1$. In these circumstances it is best to write down the solution of (4.24) and invert before using the boundary conditions to determine the constants in the solution. In view of the symmetry of the problem, it will be sufficient to consider the field for $z > 0$. Since the potential must vanish as z tends to infinity, the appropriate solution of (4.24) is

$$\bar{V} = Ae^{-pz}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.25)$$

where A is independent of z. The inversion formula (1.24) then gives

$$V=\int_0^\infty A(p)e^{-pz}pJ_0(rp)\,dp, \quad . \quad . \quad . \quad (4.26)$$

where we have written $A(p)$ in place of A to indicate that it depends on p.

$A(p)$ is determined from the "dual" integral equations obtained by inserting (4.26) in the boundary conditions (4.22). These are

$$\left.\begin{aligned}\int_0^\infty pA(p)J_0(rp)\,dp&=V_0, \quad 0\leqslant r<1,\\ \int_0^\infty p^2A(p)J_0(rp)\,dp&=0, \quad r>1.\end{aligned}\right\} \quad . \quad (4.27)$$

Dual integral equations of this type have been considered by Titchmarsh and Busbridge.* However, in the case of (4.27) the solution may be spotted from the well-known results †

$$\left.\begin{aligned}\int_0^\infty p^{-1}J_0(rp)\sin p\,dp&=\pi/2, \quad 0\leqslant r<1,\\ \int_0^\infty J_0(rp)\sin p\,dp&=0, \quad r>1.\end{aligned}\right\}$$

Hence

$$A(p)=(2/\pi)V_0p^{-2}\sin p$$

and substitution in (4.26) gives as the required solution

$$V=(2V_0/\pi)\int_0^\infty p^{-1}e^{-pz}J_0(rp)\sin p\,dp. \quad . \quad (4.28)$$

Another solution of this problem using an integral transform and oblate spheroidal coordinates is given in Chapter VI.

* E. C. Titchmarsh, *Theory of Fourier Integrals*, Oxford, (1937), p. 335. See also, I. W. Busbridge, *Proc. London Math. Soc., 2nd Series*, 44, (1938), pp. 115–129.

† G. N. Watson, *Bessel Functions*, Cambridge, (1944), p. 405.

4.4. AN APPLICATION OF THE MELLIN TRANSFORM

Two examples have already been given for an axially symmetrical stress distribution due to discontinuous surface loading. In the first (§ 3.7), the loading was on the curved surface r=constant and the solution was obtained by the complex form of the Fourier integral transform. For the second (§ 4.2), the loading was on a plane surface z=constant and the Hankel transform was used to obtain a solution. A somewhat similar problem is the determination of the stresses in an infinite wedge loaded on its plane faces and a solution for fairly general surface tractions has recently been obtained by the use of the Mellin transform.* The method is here illustrated by the following particular case of the problem.

An infinite wedge of angle 2α is subjected to unit pres-

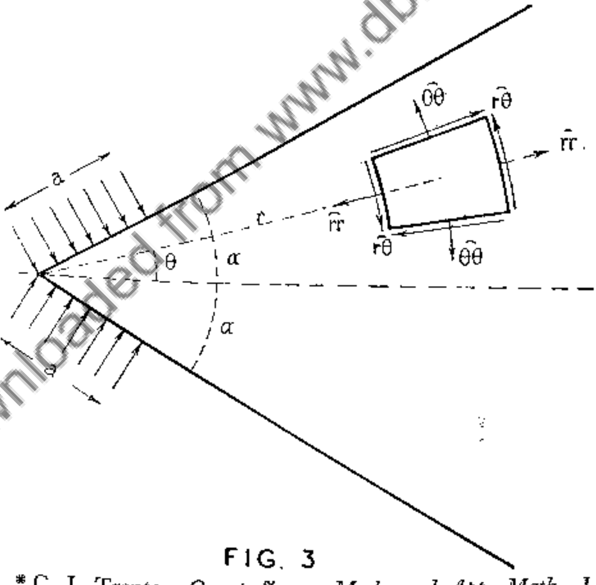

FIG. 3

* C. J. Tranter, *Quart. Journ. Mech. and App. Math.*, I, (1948), pp. 125–130.

sure on each of its plane faces for a distance a measured from the apex. It is required to find the shear stress in the wedge.

With the usual notation, illustrated in Fig. 3, the hoop and shear stresses are given in terms of a stress function χ by the relations*

$$\widehat{\theta\theta}=\partial^2\chi/\partial r^2, \quad (4\cdot29)$$

$$\widehat{r\theta}=-(\partial/\partial r)(r^{-1}\partial\chi/\partial\theta), \quad (4\cdot30)$$

and χ satisfies the biharmonic equation

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)^2\chi=0, \quad 0<r<\infty, \quad -\alpha\leqslant\theta\leqslant\alpha. \quad (4\cdot31)$$

We require the solution of equation (4·31) with the boundary conditions

$$\left.\begin{aligned}\widehat{\theta\theta}&=-1, \quad 0\leqslant r<a,\\ &=0, \quad r>a,\end{aligned}\right\}\text{ when } \theta=\pm\alpha, \quad (4\cdot32)$$

$$\widehat{r\theta}=0, \text{ when } \theta=\pm\alpha, \quad 0<r<\infty. \quad (4\cdot33)$$

Assuming χ is such that $r^{p+n}(\partial^n\chi/\partial r^n)$, $(n=0, 1, 2, 3)$, $r^p(\partial^n\chi/\partial\theta^n)$, $(n=1, 2)$ and $r^{p+1}(\partial^3\chi/\partial r\,\partial\theta^2)$ all tend to zero as r tends to infinity, and writing $\bar{\chi}$ for the Mellin transform of χ, i.e.,

$$\bar{\chi}=\int_0^\infty \chi r^{p-1}\,dr, \quad (4\cdot34)$$

integration by parts gives:

$$\left.\begin{aligned}
&\int_0^\infty r\frac{\partial^{n+1}\chi}{\partial r\,\partial\theta^n}r^{p-1}\,dr=-p\frac{d^n\bar{\chi}}{d\theta^n}, \quad n=0, 1, 2,\\
&\int_0^\infty r^2\frac{\partial^{n+2}\chi}{\partial r^2\,\partial\theta^n}r^{p-1}\,dr=p(p+1)\frac{d^n\bar{\chi}}{d\theta^n}, \quad n=0, 2,\\
&\int_0^\infty r^3\frac{\partial^3\chi}{\partial r^3}r^{p-1}\,dr=-p(p+1)(p+2)\bar{\chi},\\
&\int_0^\infty r^4\frac{\partial^4\chi}{\partial r^4}r^{p-1}\,dr=p(p+1)(p+2)(p+3)\bar{\chi}.
\end{aligned}\right\} \quad (4\cdot35)$$

* A. E. H. Love, *Mathematical Theory of Elasticity*, Cambridge, 3rd Ed., (1920), pp. 89, 202.

Multiplying (4.31) by r^{p+3}, integrating with respect to r from 0 to ∞ and using (4.35), we find

$$\frac{d^4\bar{\chi}}{d\theta^4}+[(p+2)^2+p^2]\frac{d^2\bar{\chi}}{d\theta^2}+p^2(p+2)^2\bar{\chi}=0. \quad . \quad . \quad (4.36)$$

The boundary conditions (4.32), (4.33), when use is made of (4.29), (4.30), after multiplication by r^{p+1} and integration with respect to r from 0 to ∞, yield

$$\left.\begin{aligned} p(p+1)\bar{\chi}&=-\int_0^a r^{p+1}\,dr \\ &=-(p+2)^{-1}a^{p+2}, \\ \text{and}\quad (p+1)(d\bar{\chi}/d\theta)&=0, \end{aligned}\right\} \text{ when } \theta=\pm\alpha. \quad . \quad (4.37)$$

The general solution of (4.36) is

$$\bar{\chi}=A\sin p\theta+B\cos p\theta+C\sin(p+2)\theta+D\cos(p+2)\theta,$$

where A, B, C, D depend on p and α. Since the solution is symmetrical with respect to the plane $\theta=0$, A=C=0, while (4.37) give, to determine B and C,

$$B\cos p\alpha+D\cos(p+2)\alpha=-[p(p+1)(p+2)]^{-1}a^{p+2},$$

$$Bp\sin p\alpha+D(p+2)\sin(p+2)\alpha=0.$$

These give

$$\left.\begin{aligned} B&=-\frac{a^{p+2}\sin(p+2)\alpha}{p(p+1)H(\alpha,p)}, \\ D&=\frac{a^{p+2}\sin p\alpha}{(p+1)(p+2)H(\alpha,p)}, \end{aligned}\right\} \quad . \quad . \quad . \quad (4.38)$$

where

$$H(\alpha,p)=(p+1)\sin 2\alpha+\sin 2(p+1)\alpha. \quad . \quad (4.39)$$

Hence the complete expression for $\bar{\chi}$ is

$$\bar{\chi}=\frac{-a^{p+2}}{(p+1)H(\alpha,p)}\left[\frac{\sin(p+2)\alpha\cos p\theta}{p} - \frac{\sin p\alpha\cos(p+2)\theta}{p+2}\right]. \quad . \quad (4.40)$$

Now we are only here interested in the shear stress which is given by (4.30). This can be written

$$r^2(\widehat{r\theta})=\frac{\partial\chi}{\partial\theta}-r\frac{\partial^2\chi}{\partial r\,\partial\theta},$$

and the Mellin transform of $r^2(\widehat{r\theta})$, denoted by a "bar", is

$$\overline{r^2(\widehat{r\theta})}=\int_0^\infty\left(\frac{\partial\chi}{\partial\theta}-r\frac{\partial^2\chi}{\partial r\,\partial\theta}\right)r^{p-1}\,dr$$

$$=(p+1)(d\bar{\chi}/d\theta),$$

when use is made of (4.34) and (4.35). Using (4.40) we have

$$H(\alpha,p)\overline{r^2(\widehat{r\theta})}$$

$$=a^{p+2}[\sin(p+2)\alpha\sin p\theta-\sin p\alpha\sin(p+2)\theta]. \quad (4.41)$$

The inversion formula (1.26) gives, after division by r^2,

$$\widehat{r\theta}=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\left(\frac{a}{r}\right)^{p+2}[\sin(p+2)\alpha\sin p\theta$$

$$-\sin p\alpha\sin(p+2)\theta]\frac{dp}{H(\alpha,p)}. \quad (4.42)$$

For values of α between 0 and $\frac{1}{2}\pi$ it is easy to show that the only zero of $H(\alpha,p)$ in the strip for which the real part of p lies between -2 and 0 is at $p=-1$. The line integral can therefore be replaced by integrals from $-\infty$ to 0 and from 0 to ∞ along the line for which the real part of p is -1. Omitting details of the algebra, we find

$$(\pi r/a)\widehat{r\theta}=\int_0^\infty R(p)\cos\{p\log(a/r)\}\,dp, \quad (4.43)$$

where

$$R(p)=\frac{\sin(\alpha-\theta)\sinh(\alpha+\theta)p-\sin(\alpha+\theta)\sinh(\alpha-\theta)p}{p\sin 2\alpha+\sinh 2\alpha p}. \quad (4.44)$$

For the particular case $\alpha=\frac{1}{2}\pi$, the wedge becomes a semi-infinite solid and the integral in (4.43) can be evaluated exactly by making use of the result*

$$\int_0^\infty \frac{\sinh qx}{\sinh \frac{1}{2}\pi x} \cos mx \, dx = \frac{\sin 2q}{\cos 2q + \cosh 2m}.$$

The result is

$$\widehat{r\theta} = \frac{2ar\cos\theta}{\pi}\left(\frac{a^2 \sin 2\theta}{r^4 + 2a^2 r^2 \cos 2\theta + a^4}\right),$$

which can be shown to agree with that given by Love,† who treats this particular case by an entirely different method. For other values of α it appears that the stress can only be found by evaluating the integral in (4.43) numerically. The method given in Chapter V has proved very convenient for evaluating trigonometrical integrals of this type.

The other stresses, if required, can be found similarly.

EXAMPLES ON CHAPTER IV

1. The magnetic potential Ω for a circular disc of radius a and strength ω, magnetised parallel to its axis, satisfies Laplace's equation, is equal to $2\pi\omega$ on the disc itself and vanishes at exterior points in the plane of the disc. Show that at the point (r, z), $z > 0$,

$$\Omega = 2\pi a\omega \int_0^\infty e^{-pz} J_0(rp) J_1(ap) \, dp.$$

2. Heat is supplied at a constant rate Q per unit area per unit time over a circular area of radius a in the plane $z = 0$ to an infinite solid of conductivity K. Show that the steady temperature at a point distant r from the axis of

* J. Edwards, *The Integral Calculus*, Vol. 2, Macmillan, (1922), p. 276.

† A. E. H. Love, *Phil. Trans. Roy. Soc., Series A*, 228, (1929), p. 389.

the circular area and distant z from the plane $z=0$ is given by

$$(Qa/2K)\int_0^\infty e^{-p|z|}J_0(pr)J_1(pa)p^{-1}\,dp.$$

3. Show that the shear stress at the point whose cylindrical coordinates are (r, z) in the problem of Boussinesq (§ 4.2) is given by

$$\widehat{rz}=-\int_0^\infty pze^{-pz}J_1(p)J_1(rp)\,dp.$$

4. For axially symmetrical stress, the axial and shear stresses are given in terms of a stress function χ by equations (4.1), (4.2), where χ satisfies the biharmonic equation (4.3). The axial displacement is proportional to

$$\{(1-2\sigma)\nabla^2+(\partial^2/\partial r^2)+r^{-1}(\partial/\partial r)\}\chi.$$

If a rigid circular punch of unit radius is pressed a distance d into the surface of the semi-infinite solid $z>0$, show that the axial displacement w of the surface at radial distance $r(>1)$ is given by

$$\pi w/2d=\int_0^\infty J_0(rp)p^{-1}\sin p\,dp$$
$$=\sin^{-1}(1/r).$$

5. The faces of the infinite wedge $-\alpha<\theta<\alpha$, $r>0$ are maintained at unit temperature for a distance a measured from the apex and at zero elsewhere. Use a Mellin transform to show that the steady temperature at the point (r, θ) is

$$\tfrac{1}{2}+(1/\pi)\int_0^\infty \sin\{p\log(a/r)\}\cosh p\theta\,\mathrm{sech}\,p\alpha\,p^{-1}\,dp.$$

Use the result

$$\int_0^\infty \cos mx\,\frac{\cosh qx}{\cosh\frac{1}{2}\pi x}\,dx=\frac{2\cos q\cosh m}{\cos 2p+\cosh 2m},$$

to show that the result for the special case $\alpha=\frac{1}{2}\pi$ agrees with that of Ex. 1, Chapter III.

6. In the stress problem of § 4.4, the third stress is given by

$$\widehat{rr} = \frac{1}{r}\frac{\partial \chi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \chi}{\partial \theta^2}.$$

Show that

$$\frac{\pi r}{2a}(\widehat{\theta\theta} - \widehat{rr}) = \frac{\pi \sin \alpha \cos \theta}{2\alpha + \sin 2\alpha} - \int_0^\infty P(p) \sin \{p \log (a/r)\}\, dp,$$

where

$$P(p) = \frac{\sin (\alpha - \theta) \cosh (\alpha + \theta) p + \sin (\alpha + \theta) \cosh (\alpha - \theta) p}{p \sin 2\alpha + \sinh 2\alpha p}.$$

CHAPTER V

THE NUMERICAL EVALUATION OF INTEGRALS IN SOLUTIONS

5·1. Solutions of the boundary-value problems of mathematical physics often involve infinite integrals containing a term consisting of a trigonometrical or Bessel function. Examples have been given in §§ 3·4, 3·7, 4·2, 4·3 and 4·4. Such integrals can only rarely be evaluated exactly and approximate methods have often to be used to obtain numerical values which, after all, are the ultimate aim in a physical problem.

It would appear to be useful to collect together in a single chapter methods which have proved valuable in evaluating integrals of this type. We first consider formulae which give values at the end points, usually the origin and infinity, then give some asymptotic formulae which, if applicable, are useful for moderately large values of the variable involved, and finally discuss a method (due to Filon) for the numerical evaluation of integrals containing a trigonometrical term.

5·2. VALUES AT END POINTS

We write

$$C(x)=\int_0^\infty f(p)\cos xp\,dp, \qquad (5\cdot1)$$

$$S(x)=\int_0^\infty f(p)\sin xp\,dp, \qquad (5\cdot2)$$

and

$$S_1(x)=\int_0^\infty g(p)p^{-1}\sin xp\,dp, \qquad (5\cdot3)$$

where $f(p)$, $g(p)$ are known functions of p.

Values of infinity.

If $C(\infty)$, $S(\infty)$ are used to denote the limits as x tends to infinity of $C(x)$, $S(x)$ then the Riemann-Lebesgue theorem* shows that if $|f(p)|$ is integrable over $(0, \infty)$,

$$C(\infty)=S(\infty)=0. \quad . \quad . \quad . \quad . \quad . \qquad (5.4)$$

In many cases the integral for evaluation takes the form (5.3). For this integral we have the useful result that, if $g(p)$ satisfies Dirichlet's conditions,†

$$S_1(\infty)=(\pi/2)g(0), \quad . \quad . \quad . \quad . \qquad (5.5)$$

where $S_1(\infty)$ denotes the limit of $S_1(x)$ as x tends to infinity.

As simple examples we have, $(a>0)$

$$\int_0^\infty e^{-ap} \cos xp \, dp \rightarrow 0,$$

$$\int_0^\infty e^{-ap} \sin xp \, dp \rightarrow 0,$$

and

$$\int_0^\infty e^{-ap}p^{-1} \sin xp \, dp \rightarrow \pi/2,$$

as x tends to infinity.

Values at the origin

It is immediately apparent that, if $f(p)$ is integrable over $(0, \infty)$,

$$S(0)=0, \qquad C(0)=\int_0^\infty f(p)\, dp, \quad . \quad . \qquad (5.6)$$

where $S(0)$, $C(0)$ are the limits of $S(x)$, $C(x)$ as x tends to zero.

All the above results are well known and references have been given. Not so well known is the limit of $S_1(x)$

* E. C. Titchmarsh, *Theory of Functions*, Oxford, 2nd Ed., (1939), § 13.21.

† H. S. Carslaw, *Fourier Series and Integrals*, Macmillan, 2nd Ed., (1921), § 93.

as x tends to zero. This can be obtained as follows. Let $g(p)$ be bounded, monotonic for sufficiently large p and having the limit $g(\infty)$ as x tends to infinity. Write

$$g(p)=g(\infty)+\phi(p),$$

so that $\phi(p)$ is monotonic for sufficiently large p and tends to zero as p tends to infinity. Then

$$S_1(x)=\int_0^\infty \{g(\infty)+\phi(p)\}p^{-1}\sin xp\, dp$$

$$=(\pi/2)g(\infty)+\int_0^\infty \phi(p)p^{-1}\sin xp\, dp, \qquad (5\cdot7)$$

since

$$\int_0^\infty p^{-1}\sin xp\, dp=\pi/2, \quad x>0.$$

By the second mean value theorem

$$I_1=\int_c^\infty \phi(p)p^{-1}\sin xp\, dp=\phi(c)\int_c^\alpha p^{-1}\sin xp\, dp$$

where $\alpha>c$. Hence

$$I_1=\phi(c)\int_{xc}^{x\alpha} t^{-1}\sin t\, dt<k\phi(c),$$

where k is some constant. Therefore

$$|I_1|<(\pi/2)|\phi(c)|<\varepsilon/2 \text{ for } c>c_1(\varepsilon).$$

Also if

$$I_2=\int_0^c \phi(p)p^{-1}\sin xp\, dp,$$

$$|I_2|<\int_0^c |\phi(p)p^{-1}\sin xp|\, dp$$

$$<\int_0^c |x\phi(p)|\, dp$$

$$=x\int_0^c |\phi(p)|\, dp<Mxc,$$

where M is the maximum value of $|\phi(p)|$ in the interval.

Thus $|I_2| < \varepsilon/2$ for $x < x_1(\varepsilon)$, c being fixed, and the integral on the right-hand side of (5.7) tends to zero as x tends to zero.

Hence we have the result

$$S_1(+0)=(\pi/2)g(\infty), \quad . \quad . \quad . \quad (5.8)$$

where $S_1(+0)$ denotes the limit of $S_1(x)$ as x tends to zero through positive values and $g(\infty)$ has already been defined. If x tends to zero through negative values we can show similarly that

$$S_1(-0)=-(\pi/2)g(\infty). \quad . \quad . \quad . \quad (5.9)$$

Simple examples are that, as x tends to zero, $(a>0)$,

$$\int_0^\infty e^{-ap}\cos xp\,dp \longrightarrow \int_0^\infty e^{-ap}\,dp = a^{-1},$$

$$\int_0^\infty e^{-ap}\sin xp\,dp \longrightarrow 0,$$

and that as x tends to zero through positive values

$$\int_0^\infty p\frac{\sin xp}{1+p^2}\,dp \longrightarrow \frac{\pi}{2}\lim_{p\to\infty}\left[\frac{p^2}{1+p^2}\right]=\pi/2.$$

5·3. ASYMPTOTIC FORMULAE

A method giving asymptotic series, useful for large x, for integrals of the type

$$\int_a^b f(p)F(x, p)\,dp,$$

where $F(x, p)$ is an oscillatory function such as $\sin xp$ or $J_0(xp)$, has recently been given by Willis.* We give below the substance of the method for the special cases in which we are at present interested.

* H. F. Willis, *Phil. Mag.*, 39, (1948), pp. 455–459.

Consider the integral

$$I(\alpha, x)=\int_0^\infty f(p)F(x, p)e^{-\alpha p}\, dp, \quad . \quad . \quad (5.10)$$

and suppose that $f(p)$ admits a Taylor series expansion around the origin. Assume further that the radius of convergence of this expansion covers the entire range of integration. We can therefore write

$$I(\alpha, x)=\int_0^\infty \left[\sum_{r=0}^\infty \frac{p^r}{r!} f^r(0)\right] F(x, p)e^{-\alpha p}\, dp$$

$$=\sum_{r=0}^\infty \frac{f^r(0)}{r!}\int_0^\infty F(x, p)p^r e^{-\alpha p}\, dp. \quad . \quad (5.11)$$

We impose on $F(x, p)$ the limitation that the integral

$$\int_0^\infty F(x, p)e^{-\alpha p}\, dp$$

admits of expansion in positive integral powers of α, so that

$$\int_0^\infty F(x, p)e^{-\alpha p}\, dp=\phi(\alpha), \text{ say,}$$

$$=\sum_{r=0}^\infty A_r\alpha^r, \quad . \quad . \quad (5.12)$$

where $A_r=\phi^r(0)/r!$.

If we differentiate (5.12) r times with respect to α,

$$\int_0^\infty F(x, p)p^r e^{-\alpha p}\, dp=(-1)^r\phi^r(\alpha),$$

and on substitution in (5.11) we have

$$I(\alpha, x)=\sum_{r=0}^\infty (-1)^r f^r(0)\phi^r(\alpha)/r!.$$

We now allow α to tend to zero and obtain

$$I(0, x) = \int_0^\infty f(p)F(x, p)\,dp$$

$$= \sum_{r=0}^{\infty} (-1)^r f^r(0)\phi^r(0)/r!$$

$$= \sum_{r=0}^{\infty} (-1)^r A_r f^r(0). \qquad (5.13)$$

No attempt has been made to justify the steps by which this expansion has been derived and it would be an intricate problem to obtain precise conditions under which the expansion represents the integral. Here we are content to consider particular cases on their own merits. Often the method yields an asymptotic series but sometimes the integral under discussion does not possess such an expansion and the procedure breaks down.

To sum up, if we can expand the integral

$$\int_0^\infty F(x, p)e^{-\alpha p}\,dp$$

in the form $\sum_{r=0}^{\infty} A_r \alpha^r$ then the integral

$$\int_0^\infty f(p)F(x, p)\,dp$$

will normally possess an expansion of the form

$$\sum_{r=0}^{\infty} (-1)^r A_r f^r(0).$$

Proceeding now to special cases; if we start from the known integral

$$\int_0^\infty e^{-\alpha p} \sin xp\,dp = x/(\alpha^2 + x^2)$$

$$= \frac{1}{x}\left[1 - \frac{\alpha^2}{x^2} + \frac{\alpha^4}{x^4} - \dots\right],$$

we have

$$A_{2r}=(-1)^r x^{-2r-1}, \qquad A_{2r+1}=0,$$

and hence

$$S(x)=\int_0^\infty f(p)\sin xp\,dp$$

$$=\frac{f(0)}{x}-\frac{f''(0)}{x^3}+\frac{f^{\text{iv}}(0)}{x^5}-\ldots \quad . \quad (5.14)$$

Similarly, starting from the integral

$$\int_0^\infty e^{-\alpha p}\cos xp\,dp=\alpha/(\alpha^2+x^2),$$

and expanding in ascending powers of α, we obtain

$$C(x)=\int_0^\infty f(p)\cos xp\,dp$$

$$=-\frac{f'(0)}{x^2}+\frac{f'''(0)}{x^4}-\frac{f^{\text{v}}(0)}{x^6}+\ldots \quad . \quad (5.15)$$

To obtain an asymptotic expansion for $S_1(x)$ we start with the integral

$$\int_0^\infty e^{-\alpha p}p^{-1}\sin xp\,dp=\tan^{-1}(x/\alpha)=\frac{\pi}{2}-\frac{\alpha}{x}+\frac{\alpha^3}{3x^3}-\frac{\alpha^5}{5x^5}+\ldots$$

Hence

$$A_0=\pi/2, \qquad A_{2r}=0, \qquad r>0,$$
$$A_{2r+1}=(-1)^{r+1}/(2r+1)x^{2r+1},$$

and

$$S_1(x)=\int_0^\infty g(p)p^{-1}\sin xp\,dp$$

$$=\frac{\pi}{2}g(0)+\frac{g'(0)}{x}-\frac{g'''(0)}{3x^3}+\frac{g^{\text{v}}(0)}{5x^5}-\ldots \quad . \quad (5.16)$$

In each case we have obtained an asymptotic series for the integrals. By letting x tend to infinity in (5.14), (5.15), (5.16), we obtain the limiting results given in (5.4), (5.5).

In a similar way, use of the results*

$$\int_0^\infty e^{-\alpha p} J_0(xp)\, dp = (\alpha^2+x^2)^{-\frac{1}{2}}$$

and

$$\int_0^\infty e^{-\alpha p} J_1(xp)\, dp = \{(\alpha^2+x^2)^{\frac{1}{2}}-\alpha\}x^{-1}(\alpha^2+x^2)^{-\frac{1}{2}}$$

leads to the two asymptotic formulae

$$\int_0^\infty f(p) J_0(xp)\, dp = \frac{f(0)}{x} - \frac{1}{2}\frac{f''(0)}{x^3} + \frac{1\cdot 3}{2^2 2!}\frac{f^{\text{iv}}(0)}{x^5} - \frac{1\cdot 3\cdot 5}{2^3 3!}\frac{f^{\text{vi}}(0)}{x^7} + \dots \qquad (5.17)$$

and

$$\int_0^\infty f(p) J_1(xp)\, dp = \frac{f(0)}{x} + \frac{f'(0)}{x^2} - \frac{1}{2}\frac{f'''(0)}{x^4} + \frac{1\cdot 3}{2^2 2!}\frac{f^{\text{v}}(0)}{x^6} - \frac{1\cdot 3\cdot 5}{2^3 3!}\frac{f^{\text{vii}}(0)}{x^8} + \dots \qquad (5.18)$$

5.4. FILON'S METHOD FOR THE NUMERICAL EVALUATION OF TRIGONOMETRICAL INTEGRALS

The numerical evaluation of an integral such as

$$I = \int_a^b f(p) \cos xp\, dp, \quad . \quad . \quad . \quad (5.19)$$

when x is not small, is a matter of considerable difficulty, for the ordinary quadrature formulae, such as Simpson's, require, on account of the rapid oscillation of the function $\cos xp$, the division of the range of integration into such small steps that the labour of calculation is prohibitive.

* G. N. Watson, *Bessel Functions*, Cambridge, (1944), pp. 384, 386.

Filon* has developed a special method for the evaluation of such integrals. It results in a modified form of Simpson's rule with an interval no smaller than is necessary for the numerical evaluation of the integral without the trigonometrical factor. Filon's method does not appear to be very well known and it seems worth while to give the essential parts of his paper.

Let the range be divided into $2n$ equal parts with an interval h so that

$$b=a+2nh. \quad . \quad . \quad . \quad . \quad . \qquad (5.20)$$

Write, for convenience in notation,

$$xh=\theta, \quad . \quad . \quad . \quad . \qquad (5.21)$$

$$a+sh=p_s, \quad . \quad . \quad . \quad . \qquad (5.22)$$

$$f(a+sh)=f_s, \quad . \quad . \quad . \quad . \qquad (5.23)$$

s being an integer. Suppose that over the range (p_s-h, p_s+h), that is, (p_{s-1}, p_{s+1}), the function $f(p)$ can be fitted with sufficient accuracy by the parabolic arc

$$f(p)=A+B(p-p_s)+C(p-p_s)^2. \quad . \quad . \qquad (5.24)$$

Then we find

$$\left.\begin{aligned} A&=f_s,\\ B&=(f_{s+1}-f_{s-1})/2h,\\ C&=(f_{s+1}+f_{s-1}-2f_s)/2h^2.\end{aligned}\right\} \quad . \quad . \qquad (5.25)$$

Differentiation of (5.24) with respect to p, substitution for B, C and setting $p=p_{s+1}$, p_{s-1} in turn leads to

$$\left.\begin{aligned} f'_{s+1}&=(3f_{s+1}+f_{s-1}-4f_s)/2h,\\ f'_{s-1}&=(4f_s-f_{s+1}-3f_{s-1})/2h.\end{aligned}\right\} \quad . \quad . \qquad (5.26)$$

If

$$I_s=\int_{p_{s-1}}^{p_{s+1}} f(p)\cos xp\,dp,$$

* L. N. G. Filon, *Proc. Roy. Soc. Edin.*, XLIX, (1928–29), pp. 38–47.

integration by parts gives

$$I_s=\Big[f(p)x^{-1}\sin xp\Big]_{p_{s-1}}^{p_{s+1}}-x^{-1}\int_{p_{s-1}}^{p_{s+1}}f'(p)\sin xp\,dp$$

$$=x^{-1}\Big[f(p)\sin xp+f'(p)x^{-1}\cos xp\Big]_{p_{s-1}}^{p_{s+1}}$$

$$-x^{-2}\int_{p_{s-1}}^{p_{s+1}}f''(p)\cos xp\,dp.$$

Since, from (5.24), $f''(p)=2C$, this can be written

$$xI_s=\Big[\{f(p)-2Cx^{-2}\}\sin xp+f'(p)x^{-1}\cos xp\Big]_{p_{s-1}}^{p_{s+1}}. \quad (5.27)$$

Substituting for C from (5.25), using (5.21) and remembering that

$$\sin xp_{s+1}=\sin x(p_s+h)$$
$$=\sin xp_s\cos\theta+\cos xp_s\sin\theta,$$
$$\sin xp_{s-1}=\sin x(p_s-h)$$
$$=\sin xp_s\cos\theta-\cos xp_s\sin\theta,$$

we find

$$\Big[\{f(p)-2Cx^{-2}\}\sin xp\Big]_{p_{s-1}}^{p_{s+1}}$$
$$=(f_{s+1}-f_{s-1})\sin xp_s\cos\theta$$
$$+\{(1-2\theta^{-2})(f_{s+1}+f_{s-1})+4\theta^{-2}f_s\}\cos xp_s\sin\theta. \quad (5.28)$$

Using (5.26) and working similarly with $\cos xp_s$, we have

$$\Big[f'(p)x^{-1}\cos xp\Big]_{p_{s-1}}^{p_{s+1}}$$
$$=2\theta^{-1}(f_{s+1}+f_{s-1}-2f_s)\cos xp_s\cos\theta$$
$$-\theta^{-1}(f_{s+1}-f_{s-1})\sin xp_s\sin\theta. \quad (5.29)$$

Adding (5.28), (5.29) we obtain

$$x\theta I_s=(f_{s+1}-f_{s-1})(\theta\cos\theta-\sin\theta)\sin xp_s$$
$$+(f_{s+1}+f_{s-1})(\theta\sin\theta-2\theta^{-1}\sin\theta+2\cos\theta)\cos xp_s$$
$$+4f_s(\theta^{-1}\sin\theta-\cos\theta)\cos xp_s. \quad (5.30)$$

Remembering that $xp_s = xp_{s+1} - \theta$, the coefficient of f_{s+1} in this is

$$[1+\cos^2\theta - 2\theta^{-1}\sin\theta\cos]\,\theta\cos xp_{s+1}$$
$$+[\theta+\sin\theta\cos\theta - 2\theta^{-1}\sin^2\theta]\sin xp_{s+1}.$$

Similarly by writing $xp_s = xp_{s-1} + \theta$, the coefficient of f_{s-1} in (5·30) is

$$[1+\cos^2\theta - 2\theta^{-1}\sin\theta\cos\theta]\cos xp_{s-1}$$
$$-[\theta+\sin\theta\cos\theta - 2\theta^{-1}\sin^2\theta]\sin xp_{s-1}.$$

Hence, if we write

$$\left.\begin{aligned} \theta^3\alpha &= \theta^2+\theta\sin\theta\cos\theta - 2\sin^2\theta, \\ \theta^3\beta &= 2[\theta(1+\cos^2\theta) - 2\sin\theta\cos\theta], \\ \theta^3\gamma &= 4[\sin\theta - \theta\cos\theta], \end{aligned}\right\} \quad . \quad (5{\cdot}31)$$

equation (5·30) can be written in the form, since $h=\theta/x$,

$$I_s = h[\alpha(f_{s+1}\sin xp_{s+1} - f_{s-1}\sin xp_{s-1})$$
$$+\tfrac{1}{2}\beta(f_{s+1}\cos xp_{s+1} + f_{s-1}\cos xp_{s-1}) + \gamma f_s\cos xp_s].$$

If now we sum I_s for $s=1, 3, 5, \ldots, 2n-1$, we have the formula

$$\int_a^b f(p)\cos xp\, dp = h[\alpha\{f(b)\sin xb - f(a)\sin xa\}$$
$$+\beta C_{2s} + \gamma C_{2s-1}], \quad . \quad (5{\cdot}32)$$

where C_{2s} denotes the sum of all the even ordinates of the curve $y=f(p)\cos xp$ between a and b inclusive less half the first and last ordinates, C_{2s-1} denotes the sum of all the odd ordinates, and the quantities α, β, γ are given in terms of $\theta = xh$ (h being the interval) by the relations (5·31). This formula replaces Simpson's rule for this type of integral: it holds even when x is large provided that the interval is so chosen that $f(p)$ can be fitted with reasonable accuracy by parabolic arcs.

Filon has tabulated α, β, γ when θ is given in degrees. Now that tables of the trigonometrical functions with radian argument are readily available, a similar table with θ in radians would appear to be more useful. Such a table has been computed and is given on page 71.

For small values of θ it was necessary to expand the trigonometrical terms appearing in (5.31) in order to obtain reasonable accuracy in some of the tabulated values. The resulting expansions are

$$\left.\begin{aligned}\alpha&=\frac{2\theta^3}{45}-\frac{2\theta^5}{315}+\frac{2\theta^7}{4725}-\ldots,\\ \beta&=\frac{2}{3}+\frac{2\theta^2}{15}-\frac{4\theta^4}{105}+\frac{2\theta^6}{567}-\ldots,\\ \gamma&=\frac{4}{3}-\frac{2\theta^2}{15}+\frac{\theta^4}{210}-\frac{\theta^6}{11340}+\ldots.\end{aligned}\right\} \quad . \quad . \quad (5.33)$$

When x, and therefore θ, tends to zero, these expansions show that α tends to zero, β to 2/3 and γ to 4/3. In this case the quadrature formula (5.32) becomes

$$\int_a^b f(p)\,dp=(h/3)[2C_{2s}+4C_{2s-1}],$$

which is, of course, Simpson's formula.

Table of values of α, β, γ for use with Filon's formula (5.32)

θ	α	β	γ
0·0	0·00000	0·66667	1·33333
0·025	0·00000	0·66675	1·33325
0·05	0·00001	0·66700	1·33300
0·10	0·00004	0·66800	1·33200
0·15	0·00015	0·66965	1·33034
0·20	0·00035	0·67194	1·32801
0·25	0·00069	0·67485	1·32502
0·30	0·00118	0·67836	1·32137
0·40	0·00278	0·68704	1·31212
0·50	0·00536	0·69767	1·30030
0·75	0·01730	0·73022	1·25982
1·00	0·03850	0·76526	1·20467
1·50	0·10840	0·80971	1·05646

By precisely similar analysis, the following quadrature formula for the similar integral containing a sine term can be deduced,

$$\int_a^b f(p)\sin xp\,dp = h[-\alpha\{f(b)\cos xb - f(a)\cos xa\} + \beta S_{2s} + \gamma S_{2s-1}]. \quad . \quad (5{\cdot}34)$$

Here S_{2s} denotes the sum of all the even ordinates of the curve $y=f(p)\sin xp$ between a and b inclusive less half the first and last ordinates, S_{2s-1} denotes the sum of all the odd ordinates and α, β, γ, h have their previous definitions.

When integrals of this type appear in the solution of physical problems the range of integration is usually $(0, \infty)$. For the numerical evaluation of such integrals the range can be divided into two parts $(0, \lambda)$, (λ, ∞) where λ is a suitably chosen quantity. The function $f(p)$ appearing in the integrand frequently decreases so rapidly as p increases that the integral over the second range is negligible and one of the formulae (5·32), (5·34) can be used to evaluate it for the range $(0, \lambda)$. If the integral over the range (λ, ∞) is not entirely negligible it is often possible to represent $f(p)$ with sufficient accuracy by an asymptotic series in this range and the resulting integrals can then often be evaluated in terms of the tabulated functions *

$$\left.\begin{aligned} Si(x) &= \int_0^x p^{-1}\sin p\,dp, \\ Ci(x) &= -\int_x^\infty p^{-1}\cos p\,dp. \end{aligned}\right\} \quad . \quad . \quad (5{\cdot}35)$$

The above remarks are illustrated in the example of § 5·5.

5·5. A WORKED EXAMPLE

As an illustrative example we consider the hoop stress at the surface of a long cylindrical rod when a discon-

* See, for instance, *B.A. Math. Tables*, Vol. 1, Cambridge, pp. 34–39.

tinuous pressure is applied to its curved surface. This is the problem discussed in § 3.7, and the solution is given by equations (3.46), (3.48), (3.49). Here we take Poisson's ratio, σ, of the material of the cylinder to be 0·25 and hence, writing $r=1$, the stress in question is given by

$$\widehat{\theta\theta}=-(1/2)+\pi^{-1}\int_0^\infty T(p)p^{-1}\sin zp\,dp, \quad . \quad (5.36)$$

where

$$-2T(p)=\frac{\{pI_0(p)/I_1(p)\}+3}{\{pI_0(p)/I_1(p)\}^2-p^2-1\cdot 5}. \quad . \quad (5.37)$$

As p tends to zero, $pI_0(p)/I_1(p)$ tends to 2 and hence $T(0)=-1$. Using the result (5.5) we therefore have, as z tends to infinity,

$$\int_0^\infty T(p)p^{-1}\sin zp\,dp=(\pi/2)T(0)=-\pi/2, \quad . \quad (5.38)$$

and (5.36) shows that the stress $\widehat{\theta\theta}$ tends to -1. At such a point we are far from the discontinuity in applied pressure and the result, as might be expected, is the same as that for a long cylinder under uniform pressure throughout its entire length. As z tends to minus infinity the integral in (5.38) tends to $+\pi/2$ and the hoop stress therefore tends to zero. This result is again what might be expected for here we are far from the applied load.

Use of the asymptotic series* for the Bessel functions $I_0(p)$, $I_1(p)$ in (5.37) leads to

$$-2T(p)\sim 1+4p^{-1}+1\cdot 25p^{-2}-5p^{-3}+\ \ldots \quad . \quad (5.39)$$

Hence $T(\infty)=-0\cdot 5$, and (5.8) shows that the limit as z approaches zero through positive and negative values respectively of

$$\int_0^\infty T(p)p^{-1}\sin zp\,dp=\pm(\pi/2)T(\infty)=\mp\pi/4.$$

Hence the limits in the hoop stress as we approach the

* The asymptotic expansion of $I_0(p)$ has been given in § 2.6; that for $I_1(p)$ is $\{e^p/\sqrt{(2\pi p)}\}[1-(3/8p)-(15/128p^2)-\ldots]$.

point of discontinuity in applied load from the loaded or unloaded sides are, from (5·36), respectively −0·75 and −0·25. If the other surface stresses are calculated for $z=0$, all except the shear stress (which vanishes) are discontinuous. The discontinuities are, however, such that the radial displacement of the surface of the rod is continuous for this value of z.

For other values of z, the integral was evaluated in the original investigation as follows. Above $p=12$, it was found that $T(p)$ could be represented with sufficient accuracy by the asymptotic expansion as far as the term in p^{-3} as given in (5·39). Thus

$$\widehat{\theta\theta}=-(1/2)+\pi^{-1}\int_0^{12} T(p)p^{-1}\sin zp\,dp$$
$$-(2\pi)^{-1}\int_{12}^{\infty}\{1+4p^{-1}+1\cdot 25p^{-2}-5p^{-3}\}p^{-1}\sin zp\,dp. \quad (5\cdot 40)$$

By successive integrations by parts, the second integral was evaluated as

$$\left(1-\frac{5}{8}z^2\right)\left\{\frac{\pi}{2}-Si(12z)\right\}-z\left(4+\frac{5}{6}z^2\right)Ci(12z)$$
$$+\left(\frac{3491}{10368}+\frac{5}{72}z^2\right)\sin 12z+\frac{5}{108}z\cos 12z,$$

where $Si(x)$ and $Ci(x)$ have been defined in (5·35) and have been extensively tabulated. It is thus a comparatively simple matter to evaluate the second integral in (5·40) for a given value of z.

To evaluate the first integral in (5.40), the integral

$$I=\int_0^{12}\{T(p)+1\}p^{-1}\sin zp\,dp$$

was chosen for computation by the method of § 5.4, the interval selected being $h=0\cdot 5$. The function $\{T(p)+1\}p^{-1}$ vanishes when $p=0$ and can be better fitted by parabolic arcs than $p^{-1}T(p)$ which is infinite at the origin. The first integral in (5.40) is then given by $I-Si(12z)$ and $\widehat{\theta\theta}$ can thus be completely found for a given value of z.

As a check that the substitution of the asymptotic expansion for $T(p)$ in the second integral of (5.40) did not lead to unacceptable errors, the range of integration was also divided into 0 to 10, 10 to infinity, and $\overline{00}$ was similarly computed on this basis. Little extra work was involved and excellent agreement was obtained in the two values so calculated.

EXAMPLES ON CHAPTER V

1. Show that the limits as x tends to infinity of the definite integrals

$$\text{(i)} \int_0^\infty e^{-p^2} \cos xp \, dp, \qquad \text{(ii)} \int_0^\infty \frac{\sin xp}{p(a^2+p^2)} \, dp$$

are respectively zero and $\pi/(2a^2)$.

2. Show that the limits as x tends to zero through positive values of the definite integrals

$$\text{(i)} \int_0^\infty e^{-p^2} \cos xp \, dp, \qquad \text{(ii)} \int_0^\infty \frac{p^3 \sin xp}{4a^4+p^4} \, dp$$

are respectively $\sqrt{\pi}/2$ and $\pi/2$.

3. Use the method of § 5.3 to show that

$$\int_0^\infty f(p)e^{-xp} \, dp = x^{-1}f(0)+x^{-2}f'(0)+x^{-3}f''(0)+ \dots .$$

4. If $I=\displaystyle\int_0^\infty p^{-1} \sin p \sin xp \, dp$, show that when $x>1$

$$I=x^{-1}+x^{-3}/3+x^{-5}/5+ \dots ,$$

and that when $x<1$,

$$I=x+x^3/3+x^5/5+ \dots .$$

5. The range (a, b) of the definite integral

$$I=\int_a^b f(p)e^{xp} \, dp$$

is divided into $2n$ equal parts with an interval h and $f(p)$

is fitted by parabolic arcs over ranges of width $2h$. If $\theta = xh$ and

$$\theta^3\alpha = \theta^2 + \tfrac{1}{2}\theta \sinh 2\theta + 1 - \cosh 2\theta,$$
$$\theta^3\beta = 2 \sinh 2\theta - \theta(3 + \cosh 2\theta),$$
$$\theta^3\gamma = 4\theta \cosh \theta - 4 \sinh \theta,$$

show that

$$I = h[\alpha\{f(b)e^{xb} - f(a)e^{xa}\} + \beta E_{2s} + \gamma E_{2s-1}],$$

where E_{2s} is the sum of all the even ordinates of the curve $y = f(p)e^{xp}$ less half the first and last ordinates and E_{2s-1} is the sum of all the odd ordinates of this curve.

6. The range of the definite integral $I = \int_a^b f(p) \cos xp \, dp$ is divided into n equal intervals of width h and $f(p)$ is fitted by straight lines over these intervals. If $x = \theta h$, $\alpha = \theta^{-1} - \theta^{-2} \sin \theta$, $\beta = 2\theta^{-2}(1 - \cos \theta)$, show that

$$I = h[\alpha\{f(b) \sin xb - f(a) \sin xa\} + \beta C_s]$$

where C_s is the sum of all the ordinates of the curve $y = f(p) \cos xp$ less half the first and last.

By allowing x to tend to zero, show that this formula reduces to the trapezoidal rule for numerical integration.

7. Taking an interval of 0·2, use the method of § 5.4 to evaluate numerically the integral $\int_0^\infty e^{-p^2} \cos 2p \, dp$. Compare your result with the exact value $\sqrt{\pi}/2e$.

CHAPTER VI

FINITE TRANSFORMS

6.1. So far in this monograph use has only been made of integral transforms in which the range of integration has been infinite. In our solutions of boundary-value problems we have thus only been able to exclude a variable with range $(0, \infty)$ or $(-\infty, \infty)$. It would clearly be useful to be able to employ the same technique in problems in which such a condition does not hold, that is, to use transforms to exclude a variable whose range is finite. A method of doing this was first suggested by Doetsch* for transforms with sine or cosine kernels. It has recently been extended by Sneddon† for Bessel function kernels.

The use of transforms of this type does not solve problems which are incapable of solution by the classical methods of Fourier or Fourier-Bessel series. It does, however, facilitate their solution in that the same "drill" as has already been given for transforms with an infinite range of integration can be used. In this, the use of transforms appears to have a distinct advantage over the classical methods, which often require great ingenuity in assuming at the outset the correct form of the solution.

In the succeeding paragraphs we define finite sine, cosine and Hankel transforms and obtain appropriate inversion formulae. Examples are given of the use of such transforms. Towards the end of the chapter we indicate how the method can be extended to transforms with other kernels and give an example when the kernel is a Legendre polynomial.

Limitation of space has made it necessary in this monograph to solve only comparatively simple problems as

* G. Doetsch, *Math. Annalen*, CXII, (1935), pp. 52–68.
† I. N. Sneddon, *Phil. Mag.*, 7, XXXVII, (1946), pp. 17–25.

examples. The value of the technique generally increases, however, with the complexity of the problem to be solved. This is an important point for it is the problem with, say, complicated boundary conditions which demands the greatest ingenuity in its solution by the classical method. With the present technique very little extra ingenuity is required although, of course, the algebra must be expected to be heavier in a complicated problem.

6.2. FINITE FOURIER TRANSFORMS

We define the finite sine transform by

$$\bar{f}(p)=\int_0^\pi f(x)\sin px\,dx, \quad . \quad . \quad . \quad (6.1)$$

where p is a positive integer. The choice of π as the upper limit of integration is convenient and can usually be arranged by suitable substitutions in an actual problem.

To obtain the appropriate inversion formula we make use of the ordinary theory of Fourier series. Thus if $f(x)$ can be expanded in a sine series, the coefficient a_p of $\sin px$ is given by

$$a_p=(2/\pi)\int_0^\pi f(x)\sin px\,dx$$
$$=(2/\pi)\bar{f}(p),$$

by the definition of $\bar{f}(p)$. Hence the inversion formula is

$$f(x)=(2/\pi)\sum_{p=1}^{\infty}\bar{f}(p)\sin px. \quad . \quad . \quad . \quad (6.2)$$

Similarly the finite cosine transform is defined by

$$\bar{f}(p)=\int_0^\pi f(x)\cos px\,dx, \quad . \quad . \quad . \quad (6.3)$$

where p is now a positive integer or zero. We find similarly as the inversion formula

$$f(x)=(1/\pi)\bar{f}(0)+(2/\pi)\sum_{p=1}^{\infty}\bar{f}(p)\cos px, \quad . \quad (6.4)$$

where $\bar{f}(0)$ denotes $\int_0^\pi f(x)\,dx$.

The choice of sine or cosine transform is decided by the form of the boundary conditions at the extremities of the range of the variable to be excluded. Suppose we have a term $\partial^2V/\partial x^2$ in a partial differential equation and are using a finite sine transform. We follow the usual procedure and multiply the equation by the kernel $\sin px$ and then integrate with respect to x from 0 to π. Integration by parts gives

$$\int_0^\pi \sin px(\partial^2V/\partial x^2)\,dx$$

$$=\Big[(\partial V/\partial x)\sin px\Big]_{x=0}^{\pi}-p\int_0^\pi \cos px(\partial V/\partial x)\,dx.$$

In physical problems $\partial V/\partial x$ is usually finite and therefore the first term on the right-hand side vanishes at both limits through the sine term. A second integration by parts then gives

$$\int_0^\pi \sin px(\partial^2V/\partial x^2)\,dx$$

$$=-p\Big[V\cos px\Big]_{x=0}^{\pi}-p\int_0^\pi V\sin px\,dx.$$

If suffixes 0, π are used to denote values at $x=0, \pi$ respectively and if $\bar{V}$ is used to denote the finite sine transform of V, this can be written

$$\int_0^\pi \sin px(\partial^2V/\partial x^2)\,dx=p\{V_0-(-1)^pV_\pi\}-p^2\bar{V}. \quad . \quad (6.5)$$

Similarly

$$\int_0^\pi \cos px(\partial^2 V/\partial x^2)\, dx$$

$$=(-1)^p(\partial V/\partial x)_\pi-(\partial V/\partial x)_0-p^2\bar{V}, \quad . \quad (6.6)$$

provided that V is finite when $x=0, \pi$.

The successful use of a finite sine transform in removing a term $\partial^2 V/\partial x^2$ from a differential equation therefore requires a knowledge of the values of V at each extremity of the range of x. The use of a cosine transform demands on the other hand a knowledge of the values of the derivative $\partial V/\partial x$ at the extremities of the range. Similar considerations apply in removing higher order derivatives. It should be noted, as with infinite Fourier transforms, that terms like $\partial V/\partial x$ (or any derivative of odd order) cannot be removed by these transforms for a single integration by parts leaves $\int_0^\pi V \begin{matrix}\cos\\ \sin\end{matrix} px\, dx$ in the expression for $\int_0^\pi \left(\frac{\partial V}{\partial x}\right) \begin{matrix}\sin\\ \cos\end{matrix} px\, dx$.

It sometimes happens that neither the wanted function V nor its derivative $\partial V/\partial x$ is separately specified at an extremity of the range in x. For example, when radiation takes place into a medium at zero temperature in a heat conduction problem, the boundary condition involves a linear combination of V and its derivative. In this case the boundary condition can be written

$$(\partial V/\partial x)+hV=0,$$

on $x=a$, say, where h is a constant. An appropriate transform for use with this boundary condition is easily devised. Thus we write

$$\bar{f}(p)=\int_0^a f(x) \cos px\, dx, \quad . \quad . \quad . \quad (6.7)$$

where p is not a positive integer but a positive root of the transcendental equation

$$p \tan pa=h. \quad . \quad . \quad . \quad . \quad . \quad (6.8)$$

It is easy to show that if p and q are roots of (6.8)

$$\int_0^a \cos px \cos qx \, dx = 0, \qquad p \neq q,$$

$$\int_0^a \cos^2 px \, dx = \frac{a(p^2+h^2)+h}{2(p^2+h^2)}.$$

Hence, as in ordinary Fourier series, if we can write

$$f(x) = \sum_p a_p \cos px,$$

the summation being over the positive roots of (6.8), then

$$a_p = \frac{2(p^2+h^2)}{a(p^2+h^2)+h}\int_0^a f(x) \cos px \, dx.$$

The inversion formula corresponding to the transform defined by (6.7), (6.8) is therefore

$$f(x) = 2\sum_p \frac{p^2+h^2}{a(p^2+h^2)+h} \bar{f}(p) \cos px, \qquad (6.9)$$

the summation again being over the positive roots of (6.8).

6.3. AN EXAMPLE OF THE USE OF THE FINITE SINE TRANSFORM

Suppose we wish to find a function harmonic inside the square $0<x<\pi$, $0<y<\pi$ which is constant on the edge $y=0$ and vanishes on the other edges of the square. This is the problem of the steady temperature in a long square bar when one face is kept at constant and the other faces at zero temperature. The present solution will apply to a square bar of side a if we write $\pi x/a$, $\pi y/a$ for x, y and no loss of generality occurs by working with a range 0, π.

We have to find a function V such that

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0, \qquad 0<x<\pi, \qquad 0<y<\pi, \quad (6.10)$$

with

$$V=0, \text{ when } x=0 \text{ and } \pi, \quad . \quad . \quad . \quad (6.11)$$

$$V=0, \text{ when } y=0, \quad . \quad . \quad . \quad . \quad . \quad (6.12)$$

and

$$V=V_0 \text{ (constant), when } y=\pi. \quad . \quad (6.13)$$

Since V is given when $x=0$, π we use a finite sine transform and write

$$\bar{V}=\int_0^\pi V \sin px\, dx,$$

where p is a positive integer. Since V vanishes at both ends of the range in x, (6.5) gives

$$\int_0^\pi (\partial^2 V/\partial x^2) \sin px\, dx=-p^2\bar{V}. \quad . \quad . \quad (6.14)$$

The usual procedure of multiplying the differential equation (6.10) and the remaining boundary conditions (6.12), (6.13) by the kernel $\sin px$ and integrating with respect to x between 0, π gives, when we use (6.14), the ordinary differential equation

$$(d^2\bar{V}/dy^2)-p^2\bar{V}=0, \quad . \quad . \quad . \quad (6.15)$$

and the boundary conditions

$$\bar{V}=0, \text{ when } y=0, \quad . \quad . \quad . \quad (6.16)$$

$$\bar{V}=\int_0^\pi V_0 \sin px\, dx$$

$$=-V_0\Big[(\cos px)/p\Big]_0^\pi, \text{ when } y=\pi. \quad . \quad (6.17)$$

The solution of (6.15) satisfying the first boundary condition (6.16) is

$$\bar{V}=A \sinh py.$$

The second boundary condition (6.17) shows that on $y=\pi$, $\bar{V}$ vanishes when p is even and that $\bar{V}=2V_0/p$ when p is odd. Hence $\bar{V}=0$ for p even and

$$\bar{V}=(2V_0/p) \text{ cosech } p\pi \sinh py,$$

when p is odd. The inversion formula (6.2) then gives

$$V=(4V_0/\pi)\sum_{n=0}^{\infty}(2n+1)^{-1}\operatorname{cosech}(2n+1)\pi \sinh(2n+1)y\sin(2n+1)x,$$

where we have written $p=2n+1$ for convenience in notation.

6.4. AN EXAMPLE OF THE REPEATED USE OF FINITE TRANSFORMS

Consider the similar three-dimensional problem—the steady temperature in a cube of side π when one face is at constant and the other faces are at zero temperature. This problem is governed by a partial differential equation with three independent variables and the repeated use of a finite sine transform reduces it to the solution of an ordinary differential equation.

If V is the temperature, we have to find V so that

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0, \quad . \quad . \quad . \quad (6.18)$$

inside the cube, with

$$V=V_0 \text{ (constant), when } y=\pi, \quad . \quad . \quad (6.19)$$

and so that V vanishes on the other faces $x=0$, π, $y=0$, $z=0$, π of the cube.

Writing

$$\bar{V}=\int_0^{\pi} V\sin px\,dx$$

and working as in § 6.3, the equation (6.18) and its boundary conditions transform into

$$\frac{\partial^2 \bar{V}}{\partial y^2}+\frac{\partial^2 \bar{V}}{\partial z^2}=p^2\bar{V}, \quad . \quad . \quad . \quad (6.20)$$

with, on $y=\pi$, $\bar{V}=0$ for p even, $\bar{V}=2V_0/p$ for p odd, and $\bar{V}=0$ on $y=0$, $z=0$, π.

Putting

$$\bar{V}'=\int_0^\pi \bar{V}\sin p'z\,dz$$

and operating similarly on (6.20) and its boundary conditions, we obtain the ordinary differential equation

$$d^2\bar{V}'/dy^2=(p^2+p'^2)\bar{V}', \quad . \quad . \quad . \quad . \quad (6.21)$$

with $\bar{V}'=0$ when $y=0$ and, on $y=\pi$, $\bar{V}'=0$ when p or p' is even and $\bar{V}'=4V_0/pp'$ when p and p' are both odd.

The solution of (6.21) satisfying these boundary conditions is

$$\bar{V}'=4V_0(2n+1)^{-1}(2m+1)^{-1}\operatorname{cosech} l\pi \sinh ly, \quad . \quad (6.22)$$

where

$$l^2=(2n+1)^2+(2m+1)^2, \quad . \quad . \quad . \quad (6.23)$$

and we have written $p=2n+1$, $p'=2m+1$, n and m positive integers.

Inversion to $\bar{V}$ by (6.2) gives

$$\bar{V}=\frac{8V_0}{(2n+1)\pi}\sum_{m=0}^{\infty}\frac{\sinh ly}{\sinh l\pi}\,\frac{\sin(2m+1)z}{2m+1},$$

and a further inversion to V gives as the final result

$$V=\frac{16V_0}{\pi^2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\sinh ly}{\sinh l\pi}\,\frac{\sin(2n+1)x}{2n+1}\,\frac{\sin(2m+1)z}{2m+1}. \quad (6.24)$$

6.5. AN EXAMPLE OF THE USE OF THE FINITE COSINE TRANSFORM

Suppose we wish to discuss the linear flow of heat in a solid bounded by the two parallel planes $x=0$, $x=\pi$, when its faces are thermally insulated and its initial temperature is $f(x)$. If V is the temperature at time t and κ is the diffusivity of the material of the solid, we have to solve

$$\partial V/\partial t=\kappa(\partial^2 V/\partial x^2), \qquad 0<x<\pi, \qquad t>0, \quad (6.25)$$

with the boundary conditions

$$\partial V/\partial x=0, \text{ when } x=0 \text{ and } \pi, \quad t>0, \quad . \quad (6.26)$$

and the initial condition

$$V=f(x), \text{ when } t=0, \quad 0<x<\pi. \quad . \quad . \quad (6.27)$$

In this problem the value of $\partial V/\partial x$ is given at the boundaries and we therefore use the finite cosine transform

$$\bar{V}=\int_0^\pi V \cos px\, dx,$$

p being zero or a positive integer. When account is taken of the boundary conditions (6.26), equation (6.6) shows that

$$\int_0^\pi \cos px(\partial^2 V/\partial x^2)\, dx=-p^2\bar{V}. \quad . \quad . \quad (6.28)$$

Multiplication of the governing equation (6.25) and the initial condition (6.27) by cos px, integration with respect to x between 0 and π and use of (6.28) reduces the problem to the solution of the ordinary differential equation

$$d\bar{V}/dt=-\kappa p^2\bar{V}, \quad . \quad . \quad . \quad . \quad (6.29)$$

with the condition

$$\bar{V}=\int_0^\pi f(x') \cos px'\, dx', \text{ when } t=0, \quad . \quad . \quad (6.30)$$

the dashes having been inserted to avoid confusion when we invert.

The solution of (6.29), (6.30) is clearly

$$\bar{V}=e^{-\kappa p^2 t}\int_0^\pi f(x') \cos px'\, dx',$$

and inversion by the appropriate formula (6.4) gives

$$V=(1/\pi)\int_0^\pi f(x')\, dx'$$
$$+(2/\pi)\sum_{p=1}^{\infty} e^{-\kappa p^2 t} \cos px\int_0^\pi f(x') \cos px'\, dx'. \quad (6.31)$$

6.6. AN EXAMPLE IN CONDUCTION OF HEAT WITH THE RADIATION BOUNDARY CONDITION

To illustrate the procedure when neither the wanted function nor its derivative is separately given at a boundary we consider the flow of heat in the slab $-a<x<a$ when the initial temperature has a constant value V_0, and radiation takes place at the two faces $x=\pm a$ into a medium at zero temperature. With the usual notation, the equations for solution are

$$\partial V/\partial t=\kappa(\partial^2 V/\partial x^2), \quad -a<x<a, \quad t>0, \quad (6.32)$$

with

$$\left.\begin{aligned} -(\partial V/\partial x)+hV=0, \text{ when } x=-a, \\ (\partial V/\partial x)+hV=0, \text{ when } x=a, \end{aligned}\right\} \quad . \quad . \quad (6.33)$$

and

$$V=V_0, \text{ when } t=0. \quad . \quad . \quad . \quad . \quad (6.34)$$

From the symmetry about the plane $x=0$ we can replace these by the equivalent set

$$\partial V/\partial t=\kappa(\partial^2 V/\partial x^2), \quad 0<x<a, \quad t>0, \quad . \quad (6.35)$$

with

$$\partial V/\partial x=0, \text{ when } x=0, \quad . \quad . \quad . \quad (6.36)$$

$$(\partial V/\partial x)+hV=0, \text{ when } x=a, \quad . \quad . \quad (6.37)$$

and

$$V=V_0, \text{ when } t=0. \quad . \quad . \quad . \quad . \quad (6.38)$$

Since $\partial V/\partial x$ is specified when $x=0$ the appropriate transform is one with a cosine kernel, and to deal with the condition (6.37) we take that defined by (6.7), (6.8), viz.,

$$\bar{V}=\int_0^a V \cos px \, dx, \quad . \quad . \quad . \quad (6.39)$$

where p is a positive root of the equation

$$p \tan pa=h. \quad . \quad . \quad . \quad . \quad . \quad (6.40)$$

Integration by parts gives

$$\int_0^a (\partial^2 V/\partial x^2) \cos px \, dx$$

$$= \left[(\partial V/\partial x) \cos px\right]_0^a + p\int_0^a (\partial V/\partial x) \sin px \, dx$$

$$= \left[(\partial V/\partial x) \cos px + pV \sin px\right]_0^a - p^2 \int_0^a V \cos px \, dx. \quad (6.41)$$

The first term on the right-hand side vanishes at the lower limit through (6.36). At the upper limit it can be written

$$\cos pa[(\partial V/\partial x) + pV \tan pa]_{x=a}$$

and, when use is made of (6.37), (6.40), this is seen to vanish. Thus the integral on the left of (6.41) can be replaced by $-p^2\bar{V}$, and the usual procedure applied to (6.35), (6.38), leaves for solution

$$d\bar{V}/dt = -\kappa p^2 \bar{V}, \quad . \quad . \quad . \quad . \quad (6.42)$$

with

$$\bar{V} = V_0 \int_0^a \cos px \, dx$$

$$= V_0 (\sin pa)/p, \text{ when } t=0. \quad . \quad . \quad (6.43)$$

The solution is

$$\bar{V} = V_0 e^{-\kappa p^2 t} (\sin pa/p),$$

and inversion by (6.9) gives

$$V = 2V_0 \sum_p \frac{p^2+h^2}{a(p^2+h^2)+h} \frac{\sin pa \cos px}{p} e^{-\kappa p^2 t},$$

the summation being over the positive roots of the transcendental equation (6.40). Making use of this equation, the result can be written in the slightly simpler form,

$$V = 2V_0 \sum_p \frac{h}{a(p^2+h^2)+h} \frac{\cos px}{\cos pa} e^{-\kappa p^2 t}. \quad . \quad (6.44)$$

6.7. THE FINITE HANKEL TRANSFORM

Following Sneddon, we define the finite Hankel transform by

$$\bar{f}(p)=\int_0^1 f(r)rJ_n(pr)\,dr, \quad . \quad . \quad . \quad . \quad (6.45)$$

where, for the present, p is chosen as a positive root of the equation

$$J_n(p)=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.46)$$

The choice of unity as the upper limit of the integral defining the transform is convenient and again can usually be arranged by suitable substitutions in the problem under discussion.

By the well-known theory of Fourier-Bessel series,* if p is a root of (6.46) and if $f(r)$ can be represented in the range $0 \leqslant r < 1$ by

$$f(r)=\sum_p a_p J_n(pr),$$

the coefficients a_p are given by

$$a_p=\frac{2}{J^2_{n+1}(p)}\int_0^1 f(r)rJ_n(pr)\,dr$$
$$=2\bar{f}(p)/J^2_{n+1}(p),$$

using (6.45). Hence the inversion formula for the finite Hankel transform defined as above is

$$f(r)=2\sum_p \bar{f}(p)\{J_n(pr)/J^2_{n+1}(p)\}, \quad . \quad . \quad (6.47)$$

the summation being over the positive roots of (6.46).

Other Hankel transforms can be similarly defined. Suppose, for example, we define the transform by (6.45), but, instead of taking p as a root of $J_n(p)=0$, we choose it to be a positive root of the equation

$$pJ'_n(p)+hJ_n(p)=0, \quad . \quad . \quad . \quad . \quad (6.48)$$

* G. N. Watson, *Bessel Functions*, Cambridge, (1944), p. 576.

where h is a constant. Such a transform is useful in dealing with the radiation boundary condition in the flow of heat in a circular cylinder. The appropriate inversion formula is*

$$f(r)=2\sum_p \frac{p^2\bar{f}(p)}{h^2+p^2-n^2}\,\frac{J_n(pr)}{J_n^2(p)}, \quad . \quad . \quad (6.49)$$

the summation now being over the positive roots of (6.48).

It may happen that the field of variation of the variable to be excluded does not include the origin; for example, we may be interested in the temperature or stresses within the material of a hollow tube as distinct from a solid rod. To cover this case, we take the range of variation of the variable r as (a, b) and define the transform by†

$$\bar{f}(p)=\int_a^b f(r)rB_n(pr)\,dr, \quad b>a, \quad . \quad . \quad (6.50)$$

where

$$B_n(pr)=J_n(pr)Y_n(pa)-Y_n(pr)J_n(pa), \quad . \quad (6.51)$$

and $Y_n(pr)$ is the Bessel function of the second kind of order n. We take p to be a positive root of the equation

$$J_n(pb)Y_n(pa)=Y_n(pb)J_n(pa). \quad . \quad . \quad (6.52)$$

It can then be shown that the inversion formula is

$$f(r)=\frac{\pi^2}{2}\sum_p \frac{p^2J_n^2(pb)}{J_n^2(pa)-J_n^2(pb)}\bar{f}(p)B_n(pr). \quad . \quad (6.53)$$

These transforms are useful in removing the set of terms

$$F(V)=\frac{1}{r}\,\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)-\frac{n^2V}{r^2} \quad . \quad . \quad . \quad (6.54)$$

* G. N. Watson, *Bessel Functions*, Cambridge, (1944), p. 580.

† In his original paper, Sneddon uses $G_n(pr)=-(\pi/2)Y_n(pr)$ in place of $Y_n(pr)$. Y_n is now commonly adopted as the standard second solution of Bessel's equation and its use is preferred here.

from a partial differential equation. We have, for example,

$$\int_0^1 \frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)J_n(pr)\,dr=\left[r\frac{\partial V}{\partial r}J_n(pr)\right]_0^1-p\int_0^1 r\frac{\partial V}{\partial r}J_n'(pr)\,dr.$$

We assume that V is such that the first term on the right-hand side vanishes at the lower limit. (This and the corresponding assumption in (6.55) are usually, but not always, true in physical problems. Adjustments can be made when the behaviour of V and $\partial V/\partial r$ is known when $r\to 0$.) If we choose p to be a root of $J_n(p)=0$ the first term vanishes also at the upper limit and we have

$$\int_0^1 F(V)rJ_n(pr)\,dr$$

$$=-p\int_0^1 \frac{\partial V}{\partial r}rJ_n'(pr)\,dr-n^2\int_0^1 \frac{V}{r}J_n(pr)\,dr$$

$$=-p\Big[VrJ_n'(pr)\Big]_0^1$$

$$+p\int_0^1 V\{J_n'(pr)+prJ_n''(pr)-n^2p^{-1}r^{-1}J_n(pr)\}\,dr. \quad (6.55)$$

We assume again that the first term on the right-hand side vanishes at the lower limit. If V_1 is the value of V when $r=1$ and if we use the fact that $J_n(pr)$ satisfies Bessel's equation, this can be written

$$\int_0^1 F(V)rJ_n(pr)\,dr=-pV_1J_n'(p)-p^2\bar{V}, \quad (6.56)$$

$\bar{V}$ being the Hankel transform of V defined by (6.45), (6.46).

A similar result can be obtained when p is a root of equation (6.48). For this choice of p and in the special case $n=0$, the result is

$$\int_0^1 \frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)J_0(pr)\,dr=J_0(p)\left[\frac{\partial V}{\partial r}+hV\right]_{r=1}-p^2\bar{V}, \quad (6.57)$$

where the first term means that $(\partial V/\partial r)+hV$ is to be evaluated at $r=1$.

For the transform defined by (6.50), (6.51) and (6.52) the result is

$$\int_a^b F(V) r B_n(pr)\, dr = \frac{2}{\pi}\left[V_b \frac{J_n(pa)}{J_n(pb)} - V_a\right] - p^2 \bar{V}, \quad . \quad (6.58)$$

where V_a, V_b denote values of V at $r=a$, b respectively.

Examples showing the applications of these transforms to physical problems are given in the succeeding paragraphs.

6.8. THE PROBLEM OF § 2.5. SOLVED BY A FINITE HANKEL TRANSFORM

As a first example, we apply the finite Hankel transform to determine the temperature in a long circular cylinder when its surface is kept at a constant temperature V_0, the initial temperature being zero. This is the problem solved in § 2.5 by the Laplace transform. Here we take the radius of the cylinder to be unity, so that if V is the temperature at time t and κ the diffusivity of the material, we have to solve

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} = \frac{1}{\kappa}\frac{\partial V}{\partial t}, \quad 0 \leqslant r < 1, \quad t > 0, \quad . \quad (6.59)$$

with

$$V = V_0, \text{ when } r=1, \quad t>0, \quad . \quad . \quad (6.60)$$

and

$$V = 0, \text{ when } t=0, \quad 0 \leqslant r < 1. \quad . \quad . \quad (6.61)$$

We intend to exclude r from the equation (6.59) whose left-hand side is the set of terms defined in (6.54) with $n=0$. Hence we take

$$\bar{V} = \int_0^1 V r J_0(pr)\, dr, \quad . \quad . \quad . \quad . \quad (6.62)$$

where p is a positive root of $J_0(p)=0$. By writing $n=0$ in (6.56) and remembering that V is here V_0 when $r=1$, we have

$$\int_0^1 \left(\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r}\right) r J_0(pr)\, dr = -p V_0 J_0'(p) - p^2 \bar{V}.$$

Multiplication of (6.59) and the initial condition (6.61) by $rJ_0(pr)$ and integration with respect to r between 0 and 1 therefore leaves for solution the ordinary differential equation

$$d\bar{V}/dt = -\kappa\{pV_0J_0'(p) + p^2\bar{V}\}$$

with

$$\bar{V}=0, \text{ when } t=0.$$

The differential equation can be written in the form

$$\frac{d}{dt}\left\{\bar{V}+\frac{V_0J_0'(p)}{p}\right\} = -\kappa p^2\left\{\bar{V}+\frac{V_0J_0'(p)}{p}\right\},$$

so that the appropriate solution is

$$\bar{V}=V_0p^{-1}J_0'(p)\{e^{-\kappa p^2 t}-1\}. \qquad (6.63)$$

Inverting by (6.47), since $J_0'(p)=-J_1(p)$ and n is here zero,

$$V=2V_0\sum_p\{1-e^{-\kappa p^2 t}\}\{J_0(pr)/pJ_1(p)\}, \qquad (6.64)$$

the summation being over the positive roots of $J_0(p)=0$. This can be identified with the solution given in (2.29) as follows. Since

$$\int_0^1 rJ_0(pr)\,dr = J_1(p)/p,$$

$J_1(p)/p$ is the Hankel transform of unity. Writing $n=0$, $f(p)=J_1(p)/p, f(r)=1$ in the inversion formula (6.47),

$$1=2\sum_p\{J_0(pr)/pJ_1(p)\},$$

and (6.64) can be written

$$V=V_0\left[1-2\sum_p e^{-\kappa p^2 t}\{J_0(pr)/pJ_1(p)\}\right]. \qquad (6.65)$$

By writing $p=a\alpha$ and replacing r, t by r/a, t/a^2 this result is identical with (2.29) and applies to a cylinder of radius a. The use of a finite Hankel transform in the solution of this problem has some advantage over the Laplace transform method used in § 2.5—the contour integral of the inversion formula for the Laplace transform is avoided.

6.9. HEAT FLOW IN A CYLINDER WITH RADIATION AT THE SURFACE

As an example of the use of a finite Hankel transform when p is chosen to be a root of (6.48), we find the temperature V at time t in a long cylinder of unit radius when the initial temperature is unity and radiation takes place at the surface into a medium maintained at zero temperature.

Here V is given by

$$\frac{\partial^2 V}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial r}=\frac{1}{\kappa}\frac{\partial V}{\partial t}, \quad 0 \leqslant r < 1, \quad t > 0, \quad (6.66)$$

with

$$(\partial V/\partial r)+hV=0, \text{ when } r=1, \quad t>0, \quad . \quad (6.67)$$

and

$$V=1, \text{ when } t=0, \quad 0 \leqslant r < 1, \quad . \quad . \quad (6.68)$$

where κ is the diffusivity of the material and h is a constant.

A glance at (6.57) shows that by using the transform

$$\bar{V}=\int_0^1 V r J_0(pr)\, dr,$$

choosing p to be a root of (6.48) with $n=0$, and making use of the boundary condition (6.67) we have

$$\int_0^1 \left(\frac{\partial^2 V}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial r}\right) r J_0(pr)\, dr = -p^2 \bar{V}.$$

Hence the usual procedure gives as the auxiliary equations

$$d\bar{V}/dt=-\kappa p^2 \bar{V}, \quad . \quad . \quad . \quad . \quad (6.69)$$

with

$$\bar{V}=\int_0^1 r J_0(pr)\, dr$$
$$=J_1(p)/p, \text{ when } t=0. \quad . \quad . \quad . \quad (6.70)$$

The solution is clearly

$$\bar{V}=p^{-1}J_1(p)e^{-\kappa p^2 t}, \quad . \quad . \quad . \quad . \quad (6.71)$$

and inversion by (6.49) with $n=0$ gives

$$V=2\sum_p e^{-\kappa p^2 t}\frac{pJ_1(p)}{h^2+p^2}\frac{J_0(pr)}{J_0^2(p)}, \quad . \quad . \quad (6.72)$$

where the summation is over the positive roots of the equation

$$pJ_0'(p)+hJ_0(p)=0. \quad . \quad . \quad . \quad . \quad (6.73)$$

6.10. THE MOTION OF VISCOUS FLUID BETWEEN TWO CONCENTRIC CYLINDERS

As an example of the use of the transform defined by (6.50), (6.51) and (6.52), we consider the following problem. Viscous fluid is contained between two infinitely long concentric circular cylinders of radii a and b. The inner cylinder is kept at rest and the outer cylinder suddenly starts rotating with uniform angular velocity Ω. We require to find the subsequent motion of the liquid. If v is the velocity of the fluid at time t and ν denotes the kinematical viscosity we have

$$\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}=\frac{1}{\nu}\frac{\partial v}{\partial t}, \quad a<r<b, \quad t>0, \quad . \quad (6.74)$$

with $v=\Omega b$ when $r=b$, $v=0$ when $r=a$ and $v=0$ when $t=0$.

The variable r can be removed from (6.74) by using the transform (6.50) with $n=1$. Writing $n=1$, $V_a=0$, $V_b=\Omega b$ in (6.58) we have

$$\int_a^b F(v)rB_1(pr)\,dr=\frac{2}{\pi}\Omega b\frac{J_1(pa)}{J_1(pb)}-p^2\bar{v}, \quad . \quad (6.75)$$

where

$$F(v)=\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}$$

and

$$B_1(pr)=J_1(pr)Y_1(pa)-Y_1(pr)J_1(pa). \quad . \quad (6.76)$$

Multiplication of (6.74) by $rB_1(pr)$, integration with respect to r between a and b and use of (6.75) gives

$$\frac{1}{\nu}\frac{d\bar{v}}{dt}=\frac{2}{\pi}\Omega b\frac{J_1(pa)}{J_1(pb)}-p^2\bar{v}, \quad . \quad . \quad . \quad (6.77)$$

with

$$\bar{v}=0, \text{ when } t=0. \quad . \quad . \quad . \quad . \quad (6.78)$$

The appropriate solution of (6.77) is

$$\bar{v}=\frac{2}{\pi}\frac{\Omega b}{p^2}\frac{J_1(pa)}{J_1(pb)}(1-e^{-\nu p^2 t}),$$

and inversion by (6.53) with $n=1$ leads to

$$v=\pi\Omega b\sum_p\frac{1-e^{-\nu p^2 t}}{J_1^2(pa)-J_1^2(pb)}J_1(pa)J_1(pb)B_1(pr). \quad (6.79)$$

The series giving the steady state can be summed as follows. Write $V=(r-a^2/r)$, $n=1$ in (6.58). Then it is easy to show that $F(V)=0$, $V_b=b-a^2/b$, $V_a=0$ and we have

$$\bar{V}=\frac{2}{\pi p^2}\left(\frac{b^2-a^2}{b}\right)\frac{J_1(pa)}{J_1(pb)}.$$

This function is therefore the transform of $(r-a^2/r)$ and the inversion formula (6.53) gives

$$\frac{r^2-a^2}{r}=\pi\left(\frac{b^2-a^2}{b}\right)\sum_p\frac{J_1(pa)J_1(pb)}{J_1^2(pa)-J_1^2(pb)}B_1(pr).$$

Equation (6.79) can therefore be written in the form

$$v=\frac{\Omega b^2}{r}\left(\frac{r^2-a^2}{b^2-a^2}\right)$$
$$-\pi\Omega b\sum_p\frac{J_1(pa)J_1(pb)}{J_1^2(pa)-J_1^2(pb)}B_1(pr)e^{-\nu p^2 t}, \quad . \quad (6.80)$$

the summation being over the positive roots of $B_1(pb)=0$.

6.11. EXTENSION TO OTHER KERNELS. LEGENDRE TRANSFORMS

We have seen how by a suitable choice of kernel, finite integral transforms can exclude a group of terms from a partial differential equation. Thus a Fourier sine or cosine kernel will exclude a term $\partial^2 V/\partial x^2$ while a Bessel function kerenel will exclude the set of terms

$$r^{-1}(\partial/\partial r)(r\partial V/\partial r)-(n^2V/r^2).$$

Sneddon has suggested an extension of the method by using other kernels but he gives no examples. It seems clear that transforms can be defined and inversion formulae derived when the kernels are orthogonal functions. Transforms appropriate to problems in special coordinate systems can therefore be made available and the usual "drill" followed.

We give here one example. We take a Legendre polynomial as kernel and define what may be appropriately termed a Legendre transform by

$$\bar{f}(n)=\int_{-1}^{1} f(\mu)P_n(\mu)d\mu, \quad . \quad . \quad . \quad (6.81)$$

n being a positive integer.

Since*

$$\left.\begin{aligned}\int_{-1}^{1} P_m(\mu)P_n(\mu)\,d\mu &=0, \qquad m\neq n,\\ &=2/(2n+1), \qquad m=n,\end{aligned}\right\} \quad (6.82)$$

if we assume that $f(\mu)$ can be written

$$f(\mu)=\sum_{n=0}^{\infty} a_n P_n(\mu),$$

* E. T. Whittaker and G. N. Watson, *Modern Analysis*, Cambridge, (1927), p. 305.

the coefficients a_n are given by

$$a_n = \tfrac{1}{2}(2n+1)\int_{-1}^{1} f(\mu)P_n(\mu)\,d\mu$$
$$= \tfrac{1}{2}(2n+1)\bar{f}(n).$$

The inversion theorem appropriate to the transform (6.81) is therefore

$$f(\mu) = \tfrac{1}{2}\sum_{n=0}^{\infty}(2n+1)\bar{f}(n)P_n(\mu). \quad . \quad . \quad (6.83)$$

Such a transform is of use when the range of a variable to be excluded is from -1 to $+1$. Other transforms could be set up. Thus if the range of the variable is $(0, 1)$ appropriate transforms might be

$$\bar{f}(2n+1) = \int_0^1 f(\mu)P_{2n+1}(\mu)\,d\mu, \quad . \quad . \quad (6.84)$$

and

$$\bar{f}(2n) = \int_0^1 f(\mu)P_{2n}(\mu)\,d\mu. \quad . \quad . \quad . \quad (6.85)$$

Such transforms could be conveniently termed odd and even Legendre transforms. Inversion formulae are easily found from the result*

$$\left.\begin{aligned}\int_0^1 P_m(\mu)P_n(\mu)\,d\mu &= 0, \qquad (m-n) \text{ even}, \qquad m \neq n,\\ &= 1/(2n+1), \qquad m=n,\end{aligned}\right\}$$

and are, in fact,

$$f(\mu) = \sum_{n=0}^{\infty}(4n+3)\bar{f}(2n+1)P_{2n+1}(\mu), \quad . \quad (6.86)$$

and

$$f(\mu) = \sum_{n=0}^{\infty}(4n+1)\bar{f}(2n)P_{2n}(\mu), \quad . \quad . \quad (6.87)$$

respectively.

* E. T. Whittaker and G. N. Watson, *Modern Analysis*, Cambridge, (1927), p. 306.

These transforms are useful in removing the set of terms $(\partial/\partial\mu)\{(1-\mu^2)(\partial V/\partial\mu)\}$ from an equation. Integration by parts gives

$$\int_{-1}^{1}\frac{\partial}{\partial\mu}\left\{(1-\mu^2)\frac{\partial V}{\partial\mu}\right\}P_n(\mu)\,d\mu$$
$$=\left[(1-\mu^2)\frac{\partial V}{\partial\mu}P_n(\mu)\right]_{-1}^{1}-\int_{-1}^{1}\frac{\partial V}{\partial\mu}(1-\mu^2)P_n'(\mu)\,d\mu.$$

The first term vanishes at both limits and by making use of Legendre's equation in the form

$$\frac{d}{d\mu}\{(1-\mu^2)P_n'(\mu)\}=-n(n+1)P_n(\mu),$$

we have

$$\int_{-1}^{1}\frac{\partial}{\partial\mu}\left\{(1-\mu^2)\frac{\partial V}{\partial\mu}\right\}P_n(\mu)\,d\mu$$
$$=-\left[V(1-\mu^2)P_n'(\mu)\right]_{-1}^{1}-n(n+1)\int_{-1}^{1}VP_n(\mu)\,d\mu$$
$$=-n(n+1)\bar{V}(n), \qquad (6.88)$$

where $\bar{V}(n)$ is the Legendre transform of V defined by (6.81).

The results for the odd and even transforms defined by (6.84) and (6.85) are

$$\int_{0}^{1}\frac{\partial}{\partial\mu}\left\{(1-\mu^2)\frac{\partial V}{\partial\mu}\right\}P_{2n+1}(\mu)\,d\mu$$
$$=(2n+1)P_{2n}(0)[V]_{\mu=0}-(2n+1)(2n+2)\bar{V}(2n+1), \qquad (6.89)$$

and

$$\int_{0}^{1}\frac{\partial}{\partial\mu}\left\{(1-\mu^2)\frac{\partial V}{\partial\mu}\right\}P_{2n}(\mu)\,d\mu$$
$$=-P_{2n}(0)\left[\frac{\partial V}{\partial\mu}\right]_{\mu=0}-2n(2n+1)\bar{V}(2n). \qquad (6.90)$$

It is evident that no result could be derived by using Legendre transforms which could not be obtained by a direct use of expansions in Legendre polynomials. The advantage of these transforms is that they reduce the analysis involved in the solution of boundary-value problems to a "drill" and bring the work into line with that involving Laplace, Fourier and other transforms.

6.12. THE PROBLEM OF THE ELECTRIFIED DISC SOLVED BY A LEGENDRE TRANSFORM

The Legendre transform is convenient in finding the potential due to an electrified disc, the problem discussed in § 4.3. We use oblate spheroidal coordinates (μ, ζ) related to the cylindrical coordinates (r, z) by

$$z=\mu\zeta, \qquad r=(1-\mu^2)^{\frac{1}{2}}(1+\zeta^2)^{\frac{1}{2}}. \quad . \quad . \quad (6.91)$$

The surfaces μ=constant, ζ=constant are respectively hyperboloids of revolution and oblate spheroids. The spheroid $\zeta=0$ is the circular disc $r<0$, $z=0$ while $\mu=0$ is the remainder of the plane $z=0$.

In these coordinates the differential equation (4.21) and boundary conditions (4.22) become

$$\frac{\partial}{\partial\mu}\left\{(1-\mu^2)\frac{\partial V}{\partial\mu}\right\}+\frac{\partial}{\partial\zeta}\left\{(1+\zeta^2)\frac{\partial V}{\partial\zeta}\right\}=0, \quad . \quad (6.92)$$

with

$$V=V_0, \text{ when } \zeta=0, \quad . \quad . \quad . \quad (6.93)$$

and

$$\partial V/\partial\mu=0, \text{ when } \mu=0. \quad . \quad . \quad . \quad (6.94)$$

Multiplication of (6.92) by $P_n(\mu)$, integration with respect to μ between -1 and 1 and use of (6.88) gives

$$\frac{d}{d\zeta}\left\{(1+\zeta^2)\frac{d\bar{V}(n)}{d\zeta}\right\}=n(n+1)\bar{V}(n), \quad . \quad . \quad (6.95)$$

with

$$\bar{V}(n)=V_0\int_{-1}^{1}P_n(\mu)\,d\mu, \text{ when } \zeta=0, \quad . \quad (6.96)$$

$\bar{V}(n)$ being given by

$$\bar{V}(n)=\int_{-1}^{1} VP_n(\mu)\, d\mu. \quad . \quad . \quad . \quad (6.97)$$

Now the integral on the right of (6.96) vanishes when n is a positive integer and, since $\bar{V}(n)$ tends to zero as ζ tends to infinity, we must have $\bar{V}(n)=0$ for $n>0$. For $n=0$, the integral in (6.96) is equal to 2 and $\bar{V}(0)$ satisfies

$$\frac{d}{d\zeta}\left\{(1+\zeta^2)\frac{d\bar{V}(0)}{d\zeta}\right\}=0,$$

with $\bar{V}(0)=2V_0$ when $\zeta=0$. This gives, since $\bar{V}(0)$ also tends to zero as ζ tends to infinity,

$$\bar{V}(0)=(4V_0/\pi)\cot^{-1}\zeta.$$

Putting $\bar{V}(n)=0$, $n>0$, and using the above value of $\bar{V}(0)$ in the inversion formula (6.83), the potential is given by

$$V=(2V_0/\pi)\cot^{-1}\zeta. \quad . \quad . \quad . \quad (6.98)$$

This solution can be shown to agree with the more complicated form given in (4.28) in terms of cylindrical coordinates.

The odd and even Legendre transforms defined by (6.84), (6.85), are useful in the solution of boundary-value problems connected with the semi-infinite solid $z>0$ when the boundary conditions take different forms inside and outside a circular area on the surface $z=0$. If the wanted function is specified on the surface of the solid outside the circular area, the odd transform is convenient, since the quantity $[V]_{\mu=0}$ on the right of (6.89) is known. If, on the other hand, the derivative of the wanted function is given over this area, the even transform is suitable as is seen from the corresponding formula (6.90).

EXAMPLES ON CHAPTER VI

1. (*a*) Show that the finite sine transforms of $(1-x/\pi)$ and of x/π are respectively p^{-1} and $(-1)^{p+1}\,p^{-1}$.

(*b*) Show that the finite cosine transform of

$$(\pi/3-x+x^2/2\pi)$$

is p^{-2}.

2. A string of density ρ and length π is stretched to a tension ρc^2. At time $t=0$, one end $(x=0)$ is given a small oscillation $a\sin\omega t$. If the other end remains fixed, use a finite sine transform to show that the displacement of the point x at time t is

$$a\sin\omega t\sin\omega(\pi-x)/c\operatorname{cosec}\pi\omega/c$$
$$+(2ac\omega/\pi)\sum_{p=1}^{\infty}(\omega^2-p^2c^2)^{-1}\sin px\sin pct.$$

3. For longitudinal vibrations of a uniform bar of Young's modulus E and density ρ, the stress X and displacement u at a point distant x from one end are related by $X=E(\partial u/\partial x)$. Show that $\partial^2u/\partial t^2=c^2(\partial^2u/\partial x^2)$ where $c^2=E/\rho$.

A constant force P is suddenly applied in the direction of its length to the end $(x=\pi)$ of a uniform bar of length π, mass m and unit cross-sectional area at rest on a smooth horizontal table. Use a finite cosine transform to show that the displacement at time t of the point originally at x is

$$(Pt^2/2m)+(2P/c^2m)\sum_{p=1}^{\infty}(-1)^p p^{-2}\cos px(1-\cos pct).$$

4. The cross-section of a long bar of diffusivity κ is the square $0<x<\pi$, $0<y<\pi$. If the four faces of the bar are maintained at zero temperature and the initial tempera-

ture is unity, use repeated finite sine transforms to show that the temperature at time t is

$$\phi(x).\phi(y),$$

where

$$\phi(x)=(4/\pi)\sum_{n=0}^{\infty}(2n+1)^{-1}\sin(2n+1)x\exp\{-\kappa^2(2n+1)^2t\}.$$

5. The cross-section of a long bar is the rectangle $0<x<a$, $0<y<b$. The face $y=0$ is maintained at unit temperature, there is no flow of heat over $y=b$ and $x=0$ and radiation takes place into a medium at zero temperature over $x=a$. Use the cosine transform defined by equations (6.7), (6.8), to show that the steady temperature at the point (x, y) is

$$2h\sum_p \frac{\cos px \cosh p(b-y)}{[a(p^2+h^2)+h]\cos pa \cosh pb},$$

where h is the usual constant in the radiation boundary condition.

6. If $H_n\{f(r)\}$ denotes the finite Hankel transform of $f(r)$ defined by equations (6.45), (6.46), show that, if $n>0$,

$$H_n\{r^{-1}\partial f/\partial r\}=\tfrac{1}{2}p[H_{n+1}\{r^{-1}f(r)\}-H_{n-1}\{r^{-1}f(r)\}],$$

and that

$$H_0\{r^{-1}\partial f/\partial r\}=-f(0)+pH_1\{r^{-1}f(r)\}.$$

7. A thin flexible circular membrane of unit radius and uniform surface density σ is fixed round its edge and stretched by a tension T. It is displaced symmetrically from its equilibrium position with velocity $f(r)$ and allowed to vibrate freely. Use a finite Hankel transform to show that the displacement z at time t and radial distance r is given by

$$z=\frac{2}{c}\sum_p \frac{\sin pct}{p}\,\frac{J_0(pr)}{J_1^2(p)}\int_0^1 r'f(r')J_0(pr')\,dr',$$

where $c^2=T/\sigma$ and the summation is over the positive roots of the equation $J_0(p)=0$.

8. The membrane of Ex. 7 is set in motion from rest in its equilibrium position and is subject to a uniform constant pressure P_0 acting over its whole surface for $t>0$. Show that

$$z=\frac{P_0}{T}\left\{\tfrac{1}{4}(1-r^2)-2\sum_p \frac{J_0(pr)}{p^3 J_1(p)}\cos pct\right\},$$

the summation being over the positive roots of $J_0(p)$.

9. If $B_0(pr)=J_0(pr)\,Y_0(pa)-Y_0(pr)\,J_0(pa)$ and p is a root of the equation $B_0(pb)=0$, show that

$$\int_a^b rB_0(pr)\,dr=\frac{2}{\pi p^2}\left\{\frac{J_0(pa)}{J_0(pb)}-1\right\}.$$

The two surfaces of a long cylindrical tube of internal and external radii a, b respectively are maintained at zero temperature. The diffusivity of the material of the tube is κ and its initial temperature is unity. Use the transform defined by equations (6.50), (6.51) and (6.52) with $n=0$ to show that the temperature at time t is

$$\pi\sum_p \frac{J_0(pb)B_0(pr)}{J_0(pa)+J_0(pb)}e^{-\kappa p^2 t},$$

the summation being over the positive roots of the equation $B_0(pb)=0$.

10. Solve the problem of the electrified disc (§ 6.12) by using the even Legendre transform defined by equation (6.85).

CHAPTER VII

THE COMBINED USE OF RELAXATION METHODS AND INTEGRAL TRANSFORMS

7.1. The relaxation method developed by Southwell* and his co-workers has proved very successful in the solution of boundary-value problems involving two space variables. The essentials of the method are the replacement of the partial differential equation and boundary conditions by their finite difference approximations and the approximate satisfaction of these at the nodal points of a regular (two-dimensional) network covering the region in question. When the equation involves three independent variables the finite difference approximations can still be obtained easily, but it is difficult to see how a *practical* method can be devised to satisfy these approximations at the nodal points of a *three*-dimensional network.

We have already seen how Fourier (and other) transforms can reduce the number of independent variables in a problem. If then a Fourier transform is used to remove a space variable from a three-dimensional problem it will be brought within the scope of the normal relaxational technique.

To illustrate the method, we find a function satisfying Poisson's equation inside the region bounded by a right cylinder of any cross-section when the required function is specified over the curved surface of the cylinder and when either the function itself or its normal gradient is given on the plane ends. In § 7.2 we consider cylinders of finite length and give a detailed example. In § 7.3 an outline of the solution is given for very long cylinders in which only the condition at one end affects the unknown

* R. V. Southwell, ***Relaxation Methods in Theoretical Physics***, Oxford, (1946).

function. The method could clearly be extended to problems governed by other partial differential equations.

7.2. THE FINITE CYLINDER

Take the axis of the cylinder as the z-axis and, for convenience,* let the length of the cylinder be π. Then we have to find V from

$$\nabla^2 V + f(x, y, z) = 0, \quad . \quad . \quad . \quad (7.1)$$

with

$$V = g(x, y, z), \text{ on the curved surface of the cylinder}, \quad (7.2)$$

and, if first we suppose that V is specified on the plane ends of the cylinder,

$$\left.\begin{aligned} V &= h_1(x, y), \text{ when } z = 0, \\ V &= h_2(x, y), \text{ when } z = \pi. \end{aligned}\right\} \quad . \quad . \quad . \quad (7.3)$$

Here ∇^2 is the three-dimensional Laplace operator $(\partial^2/\partial x^2)+(\partial^2/\partial y^2)+(\partial^2/\partial z^2)$ and f, g, h_1, h_2 are specified† functions of the variables indicated.

Let $\bar{V}(p)$ be the finite Fourier sine transform of V, viz.,

$$\bar{V}(p) = \int_0^\pi V \sin pz \, dz, \quad (p = 1, 2, 3, \ldots). \quad (7.4)$$

Then by (6.5),

$$\int_0^\pi \sin pz (\partial^2 V/\partial z^2) \, dz = p\{h_1 - (-1)^p h_2\} - p^2 \bar{V}(p). \quad . \quad (7.5)$$

Multiplying equations (7.1), (7.2), by $\sin pz$, integrating with respect to z between 0 and π, and using (7.5), we have

$$\nabla_1^2 \bar{V}(p) - p^2 \bar{V}(p) + F(p) = 0, \quad . \quad . \quad (7.6)$$

with

$$\bar{V}(p) = G(p), \quad . \quad . \quad . \quad . \quad (7.7)$$

* For cylinders of length c we write $\pi z/c$ in place of z.

† These functions may be specified either analytically or *numerically*.

on the bounding curve of the cross-section of the cylinder, where*

$$F(p)=p\{h_1-(-1)^p h_2\}+\int_0^\pi f \sin pz\, dz, \quad . \quad (7.8)$$

$$G(p)=\int_0^\pi g \sin pz\, dz, \quad . \quad . \quad . \quad (7.9)$$

and ∇_1^2 denotes the two-dimensional Laplace operator $(\partial^2/\partial x^2)+(\partial^2/\partial y^2)$. Thus F and G can be found in terms of the given functions f, g, h_1, h_2 as functions of x, y and p (numerical integration being used if necessary) and equations (7.6) and (7.7) specify a set of two-dimensional problems which can be solved by the usual relaxation technique.

Details of the relaxation technique can be found in Southwell's book and are not given here. Once two-dimensional maps of $\bar{V}$ have been found for integral values of p, inversion to V is given by (6.2), viz.,

$$V=(2/\pi)\sum_{p=1}^{\infty}\bar{V}(p) \sin pz. \quad . \quad . \quad (7.10)$$

The practicability of the method depends on the number of relaxation solutions of (7.6) and (7.7) with $p=1$ 2, 3, . . . required to give sufficient accuracy for calculation from (7.10). In the simple example considered below, four or five relaxation maps were sufficient for the purpose in view.

Example. Steady temperature in a cube of side π with one face at constant temperature V_0, the other faces at zero temperature.

This is the problem solved by the repeated use of sine transforms in § 6.4. The method of solution given here, although not essential in this particular example, does in fact give numerical values with little, if any, more labour

* It is assumed that f and g are such that the integrals in (7.8) and (7.9) exist.

than numerical calculation from the analytical solution (6.24).

For cylinders of irregular cross-section or cases in which any of f, g, h_1, h_2 are specified numerically, or both, an orthodox solution would probably be out of the question. Little extra difficulty would be caused in the treatment proposed here.

With the notation already employed in this section, $f=h_1=h_2=0$, $g=V_0$ when $y=\pi$, $g=0$ when $y=0$, $x=0$ and $x=\pi$. Equations (7.8) and (7.9) give $F(p)=0$,

$$G(p)=V_0\int_0^\pi \sin pz\,dz=0\ (p \text{ even}),\ =2V_0/p\ (p \text{ odd})$$

on $y=\pi$, and $G(p)=0$ on $y=0$, $x=0$, $x=\pi$.

Thus we require relaxation solutions to the problems

$$\nabla_1^2\bar{V}(2m+1)-(2m+1)^2\bar{V}(2m+1) =0, \quad (m=0, 1, 2, 3, \ldots), \quad . \quad . \quad (7.11)$$

with

$$\left.\begin{aligned}\bar{V}(2m+1)&=2V_0/(2m+1) \text{ on } y=\pi,\\ &=0 \text{ on } y=0, \quad x=0, \quad x=\pi,\end{aligned}\right\} \quad . \quad (7.12)$$

since for p even, $\bar{V}(p)$ is clearly identically zero. Relaxation maps for $m=0, 1, 2, 3$ are shown in Figs. 4–7 and, to avoid decimals, recorded values are those of

$$1000\ \bar{V}(2m+1)/V_0.$$

Reasons of symmetry make it necessary only to calculate for half the cross-section and the maps shown in the diagrams were obtained using a square mesh of side $\pi/8$ and the "difference correction" method recently published by Fox.* It should be noted that as m increases the liquidation process of the relaxation technique gets quicker and quicker.

The temperature V at any point (x, y, z) is then found by reading off values of $\bar{V}(2m+1)$ at the appropriate (x, y), multiplying by

$$(2/\pi)\sin(2m+1)z,$$

* L. Fox, *Proc. Roy. Soc., A*, 190, (1947), pp. 31–59.

2000	2000	2000	2000	
1268	1244	1148	850	0
774	743	634	395	0
459	434	352	203	0
267	250	197	110	0
150	140	109	60	0
80	74	57	31	0
34	32	25	13	0
0	0	0	0	

FIG. 4
VALUES OF $1000\overline{V}(1)/V_o$

667	667	667	667	
205	203	196	159	0
62	61	55	38	0
18	18	15	10	0
5	5	4	3	0
1	1	1	1	0
1	1	1	0	0
0	0	0	0	0
0	0	0	0	

FIG. 5
VALUES OF $1000\overline{V}(3)/V_o$

400	400	400	400	
54	54	54	47	0
7	7	7	6	0
1	1	1	1	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	

FIG. 6
VALUES OF $1000\overline{V}(5)/V_o$

286	286	286	286	
15	15	15	14	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	

FIG. 7
VALUES OF $1000\overline{V}(7)/V_o$

and summing. At the centre of the cube ($x=y=z=\pi/2$), the analytical solution (6.24) gives $V/V_0=0\cdot1668$, while the value obtained from the diagrams is

$$V/V_0=(2/\pi)[0\cdot267 \sin(\pi/2)+0\cdot005 \sin(3\pi/2)]$$
$$=0\cdot167.$$

If instead of V, the values of the normal gradient are specified on the plane ends of the cylinder, the equations for solution are (7.1), (7.2) and, in place of (7.3), the conditions

$$\left.\begin{aligned}\partial V/dz&=h_1(x, y), \text{ when } z=0,\\ &=h_2(x, y), \text{ when } z=\pi.\end{aligned}\right\} \quad . \quad (7.13)$$

We now take $\bar{V}(p)$ as the finite Fourier cosine transform of V, i.e.,

$$\bar{V}(p)=\int_0^\pi V \cos pz\, dz, \ (p=0, 1, 2, 3, \ldots). \quad . \quad (7.14)$$

Use of (6.6) and (7.13) gives

$$\int_0^\pi \cos pz\, (\partial^2 V/\partial z^2)\, dz=(-1)^p h_2-h_1-p^2\bar{V}(p), \quad . \quad (7.15)$$

and multiplication of (7.1), (7.2), by $\cos pz$, integration with respect to z from 0 to π gives equations (7.6), (7.7), where now

$$\left.\begin{aligned}F(p)&=(-1)^p h_2-h_1+\int_0^\pi f \cos pz\, dz,\\ G(p)&=\int_0^\pi g \cos pz\, dz.\end{aligned}\right\} \quad . \quad (7.16)$$

Values of $\bar{V}(p)$ are obtained for integral values (including 0) of p by the relaxation method and inversion to V is now given by

$$V=\{\bar{V}(0)/\pi\}+(2/\pi)\sum_{p=1}^{\infty}\bar{V}(p) \cos pz. \quad . \quad (7.17)$$

7.3. THE SEMI-INFINITE CYLINDER

The equations for solution are now (7.1) and (7.2) with V or $\partial V/\partial z$ specified over the flat end $z=0$ and V, $\partial V/\partial z$ tending to zero as z tends to infinity. The appropriate transforms in the two cases are now $\bar{V}(p)=\int_0^\infty V \begin{matrix}\sin\\ \cos\end{matrix}\, pz\, dz$ according as V or $\partial V/\partial z$ is specified over $z=0$. The procedure is then similar to that in § 7.2 except that all integrals are now between 0 and ∞. Two-dimensional equations analogous to (7.6) and (7.7) are obtained and relaxation solutions found for values of p sufficiently close together to permit numerical integration to give V from the inversion formulae,

$$V=\frac{2}{\pi}\int_0^\infty \bar{V}(p)\begin{matrix}\sin\\ \cos\end{matrix}\, zp\, dp.$$

EXAMPLES ON CHAPTER VII

1. Show how the finite cosine transform defined by equations (6.7), (6.8) of Chapter VI can be combined with the relaxation method to find the steady temperature in a right cylinder of finite length and any cross-section when its curved surface is maintained at a given temperature, there is no loss of heat over one of its plane ends and radiation takes place into a medium at zero temperature over the other end.

2. What modifications should be made in the solution of Ex. 1 above if, instead of one plane end of the cylinder being subject to no loss of heat, this end is maintained at zero temperature?

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